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MATHEMATICS

An Iterative Method of Solving Functional Equations

by

M. ALTMAN

Presented by W. ORLICZ on November 29, 1960

In [1] we presented an iterative method for solving non-linear functional equations in Banach spaces. This method may also be regarded as a generalization of Newton's well-known classical method. However, this generalization is essentially different from that given by L. V. Kantorovitch [2]. Besides, the Kantorovitch method requires the existence of the inverse of the Fréchet derivative and the estimate of its norm. These hypotheses present the greatest difficulty for the application of the method. On the other hand, using the Kantorovitch method we must solve a linear equation at each iteration step. All these difficulties are obviated in the method mentioned in [1]. Moreover, this method is applicable to linear operator equations, while the Kantorovitch method is not.

In connection with our method we examine here the extension of the main concept used before. In this way we obtain another generalization to Banach spaces of the well-known method of Tchebyshev. The generalization of this method to operator equations in Banach spaces has been investigated by Nečepurenko [4] and Mertvetzova [3]. However, our generalization is essentially different from the above mentioned, and its advantage is the same as in the case of our extension of Newton's method. In addition, the application of the method investigated in [4] and [3] is much more complicated than the Newton-Kantorovitch method, but this is not so in our case.

Let X be a Banach space and let $F(x)$, $x \in X$, be a non-linear continuous functional defined on the closed sphere $S(x_0, r)$ with centre x_0 and radius r.

Consider the non-linear functional equation

$$F(x) = 0. \tag{1}$$

We suppose that $F(x)$ is differentiable in the sense of Fréchet in the sphere $S(x_0, r)$ and we denote by $F(x) = F'(x)$ the Fréchet derivative of $F(x)$.

Putting $f_n = f(x_n) = F'(x_n)$, we choose a sequence of elements $y_n \in X$ such that

$$\|y_n\| = 1, \; f_n(y_n) = \|f_n\|, \; n = 0, 1, 2, \ldots, \tag{2}$$

provided that such a choice is possible. Moreover, let us assume that there exist second and third derivatives $F''(x)$ and $F'''(x)$ of $F(x)$ with bounded norms in the sphere $S(r_0, r)$.

The iterative process for solving Eq. (1) is defined as follows:

$$x_{n+1} = x_n - \left[1 + \frac{1}{2} \frac{F''(x_n) y_n^2 F(x_n)}{\|F'(x_n)^2\|}\right] \frac{F(x_n)}{\|F'(x_n)\|} y_n . \tag{3}$$

The following theorem gives the conditions of convergence of process (3), which are also sufficient conditions for the existence of a solution of Eq. (1).

THEOREM 1. *Let us assume that the following conditions are satisfied*:

1° *for the Fréchet derivative* $f_0 = F(x_0)$ *at the initial approximate solution* x_0 *the inequality*

$$1/\|f_0\| \leqslant B_0 \tag{4}$$

is fulfilled

2° *the following error estimate is known for* x_0:

$$\frac{|F(x_0)|}{\|F'(x_0)\|} \leqslant d_0 \tag{5}$$

3° *the derivatives* $F''(x)$, $F'''(x)$ *are bounded in the sphere* $S(x_0, r)$, *i.e. there exist two constants* K, L *such that*

$$\|F''(x)\| \leqslant K \text{ and } \|F'''(x)\| \leqslant L \text{ for } x \in S(x_0, r); \tag{6}$$

4° *the error estimate for the first approximate solution is known and*

$$\|x_1 - x_0\| \leqslant \left(1 + \frac{1}{2} B_0 K d_0\right) d_0 = \eta_0; \tag{7}$$

5° *the constants* B_0, K *and* η_0 *satisfy the inequality*

$$h_0 = B_0 K \eta_0 \leqslant \frac{17}{40}, \tag{8}$$

and, finally,

$$R_0 = \frac{(4 + 4h_0 + h_0^2)\frac{L}{B_0 K} + 12(1 + h_0)}{1 - h_0(1 + \frac{1}{4} h_0)} \leqslant 36,$$

$$r = \frac{40}{17} \eta_0 .$$

Then Eq. (1) *has a solution* x^* *which belongs to sphere* $S(x_0, r)$ *and the sequence of approximate solutions* x_n *defined by process* (3) *converges to* x^*. *For the error estimate we have the formula*

$$\|x_n - x^*\| \leqslant \frac{40}{17} \left(\frac{23}{40}\right)^n \left(\frac{40}{17} h_0\right)^{3^n - 1} \eta_0 . \tag{9}$$

Proof. Estimating the norm $\|f_1\| = \|F'(x_1)\|$ we get

$$\|F'(x_1)\| \geqslant \|F'(x_0)\| \left(1 - \frac{\|F'(x_0) - F'(x_1)\|}{\|F'(x_0)\|.}\right).$$

Hence, using the analogue of Lagrange's formula, we have by (4), (7) and (8)

$$\|F'(x_1)\| \geqslant \|F'(x_0)\| (1 - h_0).$$

Thus, we obtain

$$\|f_1\|^{-1} \leqslant \frac{B_0}{1 - h_0} = B_1 .$$

One can prove that

$$\frac{|F(x_1)|}{\|F'(x_1)\|} \leqslant \frac{R_0 h_0^2 \eta_0}{24 (1 - h_0)} = d_1 .$$

Hence

$$\eta_1 = \left(1 + \frac{1}{2} B_1 K d_1\right) d_1 \leqslant \left(1 + \frac{B_0 K R_0 h_0^2 \eta_0}{48 (1 - h_0)^2}\right) \frac{R_0 h_0^2 \eta_0}{24 (1 - h_0)} .$$

Thus, we get

$$\eta_1 \leqslant \left(1 + \frac{R_0 h_0^3}{48 (1 - h_0)^2}\right) \frac{R_0}{24 (1 - h_0)} h_0^2 \eta_0 ,$$

whence

$$\eta_1 \leqslant \frac{920}{289} h_0^2 \eta_0 .$$

We have

$$h_1 = B_1 K \eta_1 = \frac{B_0}{1 - h_0} K \eta_1 \leqslant \frac{1600}{289} h_0^3 \leqslant h_0 .$$

Since

$$B_1 > B_0 \text{ and } h_1 \leqslant h_0,$$

we obtain

$$R_1 = \frac{(4 + 4h_1 + h_1^2) \dfrac{L}{B_1 K} + 12 (1 + h_1)}{1 - h_1 (1 + \frac{1}{4} h_1)} < R_0 \leqslant 36 .$$

Thus, all conditions 1°—5° are satisfied for $x = x_1$ replacing the numbers B_0, d_0, η_0, h_0 and R_0 by B_1, d_1, η_1, h_1 and R_1. Hence, we can define by induction the approximate solution x_n and the corresponding numbers B_n, d_n, η_n, h_n and R_n which satisfy the following relations:

$$B_n = \frac{B_{n-1}}{1 - h_{n-1}},$$

$$\frac{|F(x_n)|}{\|F'(x_n)\|} \leqslant \frac{R_{n-1}}{24 (1 - h_{n-1})} h_{n-1}^2 \eta_{n-1} = d_n , \tag{10}$$

$$\eta_n \leqslant \frac{920}{289} h_{n-1}^2 \eta_{n-1} , \tag{11}$$

(12) $$h_n \leqslant \frac{1600}{289} h_{n-1}^3,$$

$$R_n = \frac{(4 + 4h_n + h_n^2)\dfrac{L}{B_n K} + 12(1+h_n)}{1 - h_n(1 + \frac{1}{4} h_n)} \leqslant 36.$$

In virtue of (12) we get

$$h_2 \leqslant \frac{1600}{289} h_1^3 \leqslant \frac{1600}{289}\left(\frac{1600}{289} h_0^3\right)^3 = \frac{17}{40}\left(\frac{40}{17} h_0\right)^{3^2}, \ldots$$

and

$$h_n \leqslant \frac{17}{40}\left(\frac{40}{17} h_0\right)^{3^n}.$$

It follows from (11) and (12) that

(13) $$\eta_n \leqslant \left(\frac{23}{40}\right)^n \left(\frac{40}{17} h_0\right)^{3^n - 1}.$$

We shall now prove the convergence of the sequence of approximate solutions x_n. We have

$$\|x_{n+p} - x_n\| \leqslant \eta_n + \eta_{n+1} + \ldots + \eta_{n+p-1}.$$

Hence, we get by (13)

$$\|x_{n+p} - x_n\| \leqslant \left(\frac{23}{40}\right)^n \left(\frac{40}{17} h_0\right)^{3^n - 1} \eta_0 + \left(\frac{23}{40}\right)^{n+1} \left(\frac{40}{17} h_0\right)^{3^{n+1} - 1} \eta_0 +$$

$$+ \ldots + \left(\frac{23}{40}\right)^{n+p-1} \left(\frac{40}{17} h_0\right)^{3^{n+p-1} - 1} \eta_0 \leqslant$$

$$\leqslant \left(\frac{23}{40}\right)^n \left(\frac{40}{17} h_0\right)^{3^n - 1} \eta_0 \left[1 + \frac{23}{40} + \ldots + \left(\frac{23}{40}\right)^{p-n}\right],$$

(14) $$\|x_{n+p} - x_n\| \leqslant \frac{40}{17}\left(\frac{23}{40}\right)^n \left(\frac{40}{17} h_0\right)^{3^n - 1} \eta_0 \left[1 - \left(\frac{23}{40}\right)^p\right].$$

Thus, we infer that there exists an element x^* such that $x_n \to x^*$ and

$$\|x^* - x_n\| \leqslant \frac{40}{17}\left(\frac{23}{40}\right)^n \left(\frac{40}{17} h_0\right)^{3^n - 1} \eta_0.$$

Putting $n = 0$ in (14) we obtain

$$\|x_n - x_0\| \leqslant \frac{40}{17} \eta_0 \quad \text{for} \quad n = 1, 2, \ldots$$

and

$$\|x^* - x_0\| \leqslant \frac{40}{17} \eta_0.$$

It remains to prove that x^* is a solution of Eq. (1). Using inequality (10) we have

$$\frac{|F(x_n)|}{\|F'(x_n)\|} \to 0 \quad \text{as} \quad n \to \infty.$$

Since the sequence of $\|F'(x_n)\|$ is bounded, we get

$$F(x_n) \to 0 \text{ as } n \to \infty.$$

It follows from the continuity of $F(x)$ that

$$F(x^*) = 0.$$

Remark. Theorem 1 remains true, if one replaces condition (8) by the following

$$h_0 = B_0 K \eta_0 \leqslant \frac{2}{5}. \tag{15}$$

In this case we put $r = 5/2\,\eta_0$ and the error estimate will be given by the formula

$$\|x^* - x_n\| \leqslant \frac{5}{2}\left(\frac{3}{5}\right)^n \left(\frac{5}{2} h_0\right)^{3^n - 1} \eta_0 \tag{16}$$

instead of (9).

Thus, we can see that the hypotheses of Theorem 1 give a slightly better convergence. The same remark is also true for the operator variant of the Tchebyshev method investigated in the paper of Mertvetzova [3], where the hypothesis corresponding to (15) gives the error estimate (16).

We shall now give an application of Theorem 1 to operator equations in Hilbert spaces.

Let H be a real Hilbert space and let $S(x_0, r)$ be a closed sphere of H. Consider the non-linear operator equation

$$P(x) = 0, \tag{16}$$

where $P(x)$ is a non-linear continuous operator defined on the sphere $S(x_0, r)$ with values in H. Let us ussume that $P(x)$ is three times differentiable in the sense of Fréchet and denote by $P'(x)$, $P''(x)$ and $P'''(x)$ the first, second and third Fréchet derivative of $F(x)$, respectively. Suppose also that the derivatives $P''(x)$ and $P'''(x)$ are bounded, on the sphere $S(x_0, r)$.

Operator equation (16) can be reduced to functional equation (1) as follows. Putting, for instance,

$$F(x) = \|P(x)\|^2 = 0 \tag{17}$$

we see that Eqs. (16) and (17) are equivalent. Process (3) can here be defined as follows. First of all, we have to calculate the derivatives $F'(x)$ and $F''(x)$. Thus, we get

$$F'(x) = 2\,Q(x),$$

where $Q(x) = \overline{P'(x)}\,P(x)$ and the linear operator $\overline{P'(x)}$ is the adjoint of $P'(x)$. Using the differentiation rules we obtain

$$F''(x)\,\Delta x\,\Delta_1 x = 2\,(P'(x)\,\Delta_1 x, P'(x)\,\Delta x) + 2\,(P(x), P''(x)\,\Delta_1 x \Delta x).$$

Putting

$$y = \frac{Q(x)}{\|Q(x)\|},$$

we get

$$F''(x)\,y^2 = 2\,\|P'(x)\,Q(x)\|^2 + 2\,(P(x),\,P''(x)\,[Q(x)]^2).$$

Hence, process (3) yields

$$(18)\qquad x_{n+1} = x_n - \frac{1}{2}\left[1+\frac{\|P'(x_n)Q(x_n)\|^2+(P(x_n),\,P''(x_n)[Q(x_n)]^2)}{4\,\|Q(x_n)\|^4}\,F(x_n)\right]\frac{F(x_n)}{\|Q(x_n)\|^2}\,Q(x_n).$$

Applying Theorem 1 we get the following

THEOREM 2. *Let us assume that the following conditions are satisfied*:

1) $$\frac{1}{\|Q(x_0)\|} \leqslant B_0,$$

2) $$\frac{\|P(x_0)\|^2}{\|Q(x_0)\|} \leqslant d_0,$$

3) *there exist two constants* K, L *such that*

$$\|F''(x)\| \leqslant K \text{ and } \|F'''(x)\| \leqslant L \text{ for } x \in S(x_0, r),$$

where $F(x)$ *is defined in* (17);

4) $$\|x_1 - x_0\| \leqslant (1+\tfrac{1}{8}\,B_0\,Kd_0)\,d_0 = \eta_0;$$

5) $$h_0 = B_0\,K\eta_0 \leqslant 1,7;$$

$$R_0 = \frac{\left(4+h_0+\frac{1}{16}\,h_0^2\right)\frac{L}{B_0K}+6\left(1+\frac{1}{4}\,h_0\right)}{1-\frac{1}{4}\,h_0\,(1+h_0)} \leqslant 18;$$

$$r = \frac{20}{17}\,\eta_0.$$

Then Eq. (16) *has a solution* x^* *which belongs to sphere* $S(x_0, r)$ *and the sequence of approximate solutions* x_n *defined by process* (18) *converges toward* x^*. *The error estimate is given by the formula*

$$\|x_n - x^*\| \leqslant \frac{20}{17}\left(\frac{23}{40}\right)^n\left(\frac{10}{17}\,h_0\right)^{3^n-1}\eta_0.$$

We can obtain different conditions of convergence of process (18) using different estimates for the derivatives in (3).

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY, PAN)

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MATHEMATICS

Iterative Methods of Higher Order

by

M. ALTMAN

Presented by W. ORLICZ on November 29, 1960

A second order iterative method of solving functional equations in Banach spaces was given in [1]. This method is actually a generalization of Newton's well-known classical method, but essentially different from that given by L. V. Kantorovitch [5]. A third order iterative method of solving functional equations in Banach spaces is given in [3]. This method is a generalization of the well-known method of Tchebyshev. In this note we shall consider another class of iterative methods of higher order for solving functional equations in Banach spaces. The formalism of these methods is based on the application of König's theorem. In the case of the second order iterative methods of this class we obtain again Newton's method. The third order iterative methods of this class coincide with the methods of tangent hyperbolas investigated by Salehov [8] and Mertvetzova [7]. The argument used here is actually a further extension of the main idea used in paper [1]. An examination of our variant of the method of tangent hyperbolas is given by using the majorant principle. This principle has been used by Kantorovitch [6] in order to examine Newton's method. An application of the same principle to an examination of our method is given in paper [2].

Let $f(x)$ be a function, where x and $f(x)$ are numbers. Consider the equation

$$f(x) = 0. \tag{1}$$

We shall use the following known iterative method for solving Eq. (1). Let us put formally

$$c_p(x) = \frac{1}{p!}\left[\frac{1}{f(t)}\right]^{(p)}_{t=x},$$

where the upper index means the derivative of order p. Then the iterative method of order $p+2$ is defined by the following formula

$$x_{n+1} = g_p(x_n) \qquad (n = 0, 1, 2, \ldots), \tag{2}$$

where

$$g_p(x) = x + \frac{c_p(x)}{c_{p+1}(x)}$$

or

$$g_p(x) = x + (p+1)\frac{\left[\frac{1}{f(t)}\right]^{(p)}_{t=x}}{\left[\frac{1}{f(t)}\right]^{(p+1)}_{t=x}}.$$

Putting in (2) $p = 0$ we obtain Newton's classical method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{3}$$

For $p = 1$, formula (2) gives the method of tangent hyperbolas;

$$x_{n+1} = x_n - \frac{1}{1 - \frac{1}{2}\frac{f''(x_n)f(x_n)}{[f'(x_n)]^2}}\frac{f(x_n)}{f'(x_n)}. \tag{4}$$

This method is of order three. For $p = 2$ formula (3) gives

$$x_{n+1} = x_n + \frac{1 - \frac{1}{2}f''(x_n)\frac{f(x_n)}{[f'(x_n)]^2}}{1 - f''(x_n)\frac{f(x_n)}{f'(x_n)} + \frac{1}{6}f'''(x_n)\left[\frac{f(x_n)}{f'(x_n)}\right]^2}\cdot\frac{f(x_n)}{f'(x_n)}. \tag{5}$$

It is known that the expressions for $c_p(x)$ can be obtained from the following recurrent relations

$$\begin{aligned}
1 &= a_0(x)\ c_0(x),\\
0 &= a_1(x)\ c_0(x) + a_0(x)\ c_1(x),\\
0 &= a_2(x)\ c_0(x) + a_1(x)\ c_1(x) + a_0(x)\ c_2(x),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
0 &= a_p(x)\ c_0(x) + a_{p-1}(x)\ c_1(x) + \ldots + a_0(x)\ c_p(x),
\end{aligned}$$

where

$$a_k(x) = \frac{1}{k!}f^{(k)}(x).$$

Let X be a Banach space and let $F(x)$, $x \in X$, be a non-linear continuous functional, defined on the closed sphere $S(x_0, r)$ with centre x_0 and radius r.

Consider the non-linear functional equation

$$F(x) = 0. \tag{6}$$

We suppose that $F(x)$ is differentiable in the sense of Fréchet in the sphere $S(x_0, r)$ and we denote by $F'(x)$, $F''(x)$, $F'''(x)$, ... the first, second, third, ... derivatives of $F(x)$.

For the sequence of linear functionals $F'(x_n)$ we choose a sequence of elements y_n of X such that

$$\|y_n\| = 1 \quad F'(x_n)\,y_n = \|F'(x_n)\|, \quad n = 0, 1, 2$$

provided that such a choice is possible.

We shall now construct the iterative methods for solving Eq. (6). For this purpose we use formula (2). Thus, we obtain the following general iterative procedure

$$x_{n+1} = x_n + \frac{c_{p+1}(x_n)}{c_p(x_n)} y_n,$$

where $f(x_n)$, $f'(x_n)$, $f''(x_n)$, ..., $f^{(k)}(x_n)$ should be replaced by $F(x_n)$, $F'(x_n)\, y_n$, $F''(x_n)\, y_n^2$, ..., $F^{(k)}(x_n)\, y_n^k$, respectively.

Thus, for $p = 0$ we obtain the following generalization of Newton's method presented in [1]

$$x_{n+1} = x_n - \frac{F(x_n)}{\|F'(x_n)\|} y_n.$$

For $p = 1$ we obtain the following generalization of the method in (4)

$$x_{n+1} = x_n - \frac{1}{1 - \frac{1}{2} \frac{F''(x_n)\, y_n^2\, F(x_n)}{\|F'(x_n)\|^2}} \frac{F(x_n)}{\|F'(x_n)\|} y_n. \tag{7}$$

For $p = 2$ we get the following generalization of the method in (5)

$$x_{n+1} = x_n + \frac{1 - \frac{1}{2} F''(x_n)\, y_n^2 \frac{F(x_n)}{\|F'(x_n)\|^2}}{1 - F''(x_n)\, y_n^2 \frac{F(x_n)}{\|F'(x_n)\|} + \frac{1}{6} F'''(x_n)\, y_n^3 \frac{F^2(x_n)}{\|F'(x_n)\|^2}} \frac{\|F'(x_n)\|}{F'(x_n)} y_n.$$

We shall now apply the majorant principle to the examination of the iterative method defined by formula (7).

Thus, we consider the real equation

$$Q(z) = 0, \tag{8}$$

where $Q(z)$ is a real function of the real variable z, being three times continuously differentiable in the interval (z_0, z').

Following [1] let us say that Eq. (6) has a real majorant equation (8), if the following conditions are satisfied.

1° $$|F(x_0)| \leqslant Q(z_0), \qquad z_0 \geqslant 0;$$

2° $$\frac{1}{\|F'(x_0)\|} \leqslant B_0,$$

where $B_0 = -\frac{1}{Q'(z_0)} > 0$;

3° $$\|F''(x)\| \leqslant Q''(z) \text{ if } \|x - x_0\| \leqslant z - z_0 \leqslant z' - z_0;$$

4° $$\|F'''(x)\| \leqslant Q'''(z) \text{ if } \|x - x_0\| \leqslant z - z_0 \leqslant z' - z_0.$$

The following simple lemma will be applied:

LEMMA. *If the sign of the product* $Q'''(z)\,Q(z)$ *is constant in* (z_0, z^*) *and* $[Q'(z_0)]^2 - 2Q''(z_0)\,Q(z_0) > 0$, *then*

$$\frac{Q''(z)\,Q(z)}{[Q'(z)]^2} \leqslant \frac{1}{2} \quad \text{for} \quad z_0 \leqslant z \leqslant z^*,$$

where z^* *is a root of* Eq. (8), *i.e.* $Q(z^*) = 0$.

The following theorem gives sufficient conditions of the convergence of process (7) which are also sufficient for the existence of a solution of Eq. (6).

THEOREM. *Suppose that* Eq. (6) *has a majorant* Eq. (8) *such that*

$$[Q'(z_0)]^2 - 2Q''(z_0)\,Q(z_0) > 0.$$

Eq. (8) *has a positive solution and*

$$Q'(z) < 0 \quad \text{for} \quad z_0 \leqslant z \leqslant \frac{7}{3}z^*, \tag{9}$$

where z^* *is the smallest positive root of* Eq. (8). *Then process* (4) *defined for* $Q(z)$ *converges to* z^*, *and* Eq. (6) *has a solution* x^*, *to which process* (7) *converges, i.e.* $x_n \to x^*$. *The error estimate is given by the following formula*

$$\|x^* - x_n\| \leqslant z^* - z_n,$$

where z_n *is the approximate solution of* Eq. (8) *defined by process* (4).

Proof. It follows from the Lemma and from the conditions of the Theorem that

$$\|x_1 - x_0\| \leqslant z_1 - z_0. \tag{10}$$

We shall now show that all conditions 1°—4° are satisfied, if element x_0 is replaced by x_1. Using the analogue of Taylor's formula in the integral form we get

$$F(x_1) = F(x_0) + \frac{F(x_0)}{1 - \frac{1}{2}\dfrac{F''(x_0)\,y_0^2\,F(x_0)}{\|F'(x_0)\|^2}} +$$

$$+ \frac{1}{2}F''(x_0)\,y_0^2 \frac{1}{\left[1 - \dfrac{1}{2}\dfrac{F''(x_0)\,y_0^2\,F(x_0)}{\|F'(x_0)\|^2}\right]^2} \frac{[F(x_0)]^2}{\|F'(x_0)\|^2} + \frac{1}{6}\int_{x_0}^{x_1} F'''(\bar{x})\,(x_1 - x)^3\,dx\,.$$

Since a similar formula is true for $Q(z_1)$, we get

$$|F(x_1)| \leqslant Q(z_1).$$

Since $Q''(z) \geqslant 0$, the derivative $Q'(z)$ increases, still preserving the minus sign at point z_1. Further, we obtain

$$\|F'(x_1)\| \geqslant \|F'(x_0)\| \left(1 - \frac{\|F'(x_1) - F'(x_0)\|}{\|F'(x_0)\|}\right).$$

Using the abstract analogue of the fundamental formula of the integral calculus, we obtain by 3° and (10)

$$\|F'(x_1)\| \geqslant \|F'(x_0)\| \left(1 - \frac{\left\| \int_{x_0}^{x_1} F''(x)\,dx \right\|}{\|F'(x_0)\|}\right) \geqslant \|F'(x_0)\| \left(1 - \frac{\int_{z_0}^{z_1} Q''(z)\,dz}{\|F'(x_0)\|}\right) \geqslant$$

$$\geqslant \|F'(x_0)\| \left(1 + \frac{Q'(z_1) - Q'(z_0)}{Q'(z_0)}\right) = \|F'(x_0)\| \frac{Q'(z_1)}{Q'(z_0)}.$$

Hence, we get

$$\frac{1}{\|F'(x_1)\|} \leqslant \frac{1}{Q'(z_1)}.$$

It is easy to see that conditions 3° and 4° are also satisfied for x_1. In fact, if

$$\|x - x_1\| \leqslant z - z_1 \leqslant z' - z_1,$$

then we have, by (10),

$$\|x - x_0\| \leqslant \|x - x_1\| + \|x_1 - x_0\| \leqslant z - z_1 + z_1 - z_0 = z - z_0 \leqslant z' - z_0.$$

In the same way as above we get

$$\|x_2 - x_1\| \leqslant z_2 - z_1,$$

and we obtain by induction

$$\|x_{n+p} - x_n\| \leqslant z_{n+p} - z_n.$$

Suppose that the sequence of z_n converges to z^*. Hence, we infer that there exists an element x^* such that the sequence of x_n converges to x^* and

$$\|x_n - x^*\| \leqslant z_n - z^*.$$

Using the Lemma we get, by (7)

$$\frac{F(x_n)}{\|F'(x_n)\|} \to 0.$$

Hence, it follows that x^* is a solution of Eq. (6). It remains to prove that the sequence of z_n converges to z^*. Since this sequence is increasing, it is sufficient to show that it is bounded. In virtue of the lemma using a simple argument one can show that

$$z_n \leqslant \frac{7}{3} z^*, \quad n = 0, 1, 2, \ldots,$$

where z^* is the smallest positive solution of Eq. (8).

Remark. Condition (9) can be replaced by the following one

$$Q'(z_n) < 0, \quad n = 0, 1, 2, \ldots$$

A convergence theorem for process (7) has been stated without proof by B. Janko [4].

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
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MATHÉMATIQUE

Transformations des intégrales itérées singulières dans l'espace

par

W. POGORZELSKI

Présenté par T. WAŻEWSKI le 6 décembre, 1960

Les intégrales itérées singulières (au sens de Cauchy) dans l'espace ont été étudiées par Tricomi [1], Giraud [2] et Mikhline [3]. Dans le présent travail nous allons étudier ces intégrales sous d'autres hypothèses, en nous appuyant sur les propriétés des fonctions de classe $\mathfrak{H}_\alpha^h$ introduites et étudiées dans le travail [4].

Soit dans l'espace euclidien à n dimensions ($n \geqslant 2$) un ensemble fini de $p+1$ surfaces fermées de Liapounoff $S_0, S_1, \ldots, S_p$ à $n-1$ dimensions ($p \geqslant 0$). On admet que ces surfaces sont disjointes et que les surfaces $S_1, \ldots, S_p$ sont situées de façon arbitraire à l'intérieur du domaine borné Ω_0 limité par la surface S_0.*)

Désignons par Ω un ensemble de tous les points du domaine Ω_0 non situés sur les surfaces $S_1, S_2, \ldots, S_p$. Cet ensemble est une somme $\Omega = \Omega_1 + \Omega_2 + \ldots + \Omega_m$ de domaines disjoints, simplement ou multiplement connexes. En cas d'absence des surfaces $S_1, \ldots, S_p$ on a $\Omega = \Omega_0$.

Soit $G(x)$ une fonction réelle ou complexe, définie en tout point $x \neq (0, 0, \ldots 0)$ de l'espace par la formule

$$G(x) = \frac{g(x')}{|x|^n} \tag{1}$$

où $x' = \frac{x}{|x|}$ est la projection centrale du point x sur la surface ω d'une sphère, centrée en $(0, 0, \ldots 0)$, de rayon unité. On admet que la fonction $g(x')$, définie sur la surface ω, vérifie l'égalité intégrale

$$\int_\omega g(x')\, dx' = 0 \tag{2}$$

et satisfait à la condition de Hölder

$$|g(x') - g(x'')| < k_\omega |x' - x''|^{h_\omega} \tag{3}$$

($k_\omega > 0, 0 < h_\omega \leqslant 1$), où $|x' - x''|$ désigne la distance des points x' et x''.

*) Nos considérations s'appliquent aussi au cas où le domaine Ω_0 est multiplement connexe, limité par un ensemble de surfaces fermées $S_0^{(1)}, S_0^{(2)}, \ldots, S_0^{(r)}$.

Soit ensuite une fonction réelle ou complexe $K(x, y)$, à faible singularité pour $x = y$, définie par la formule

$$K(x, y) = \frac{k(x, y)}{|x - y|^{\gamma}}; \quad (0 \leqslant \gamma < n) \tag{4}$$

où la fonction $k(x, y)$, définie pour $x \in \Omega_0 + S_0$, $y \in \Omega_0 + S_0$, vérifie une inégalité de Hölder

$$|k(x, y) - k(x_1, y_1)| < \text{Const}\,[|x - x_1|^{h_0} + |y - y_1|^{h_0}] \tag{5}$$

$(0 < h_0 \leqslant 1)$, $|x - y|$ etc. désigne la distance euclidienne des points x et y. Les propriétés de transformation des intégrales itérées singulières, étendues au domaine Ω, sont exprimées par les deux théorèmes suivants.

Théorème 1. *Si la fonction $f(x)$, définie dans la région Ω ou Ω_0, appartient à la classe $\mathfrak{H}_{\alpha}^{h}$, où $h < h_{\omega}$, $\alpha > 0$, $\alpha + h < 1$, donc, elle vérifie les inégalités*

$$\begin{aligned} |f(x)| &< \frac{\text{Const}}{|x - x|^{\alpha}} \\ |f(x) - f(\tilde{x})| &< \frac{\text{Const}\ |x - \tilde{x}|^{h}}{|x - x_S|^{\alpha + h}} \end{aligned} \tag{5'}$$

(x_S désignant un point des surfaces $S_0, \ldots, S_p$ le plus rapproché du point x, on admet $|x - x_S| \leqslant |\tilde{x} - \tilde{x}_S|$); [4], alors l'égalité des intégrales itérées suivantes est vraie

$$\int_{\Omega} K(x, z)\Big[\int_{\Omega} G(z - y) f(y)\, dy\Big]\, dz = \int_{\Omega} \Big[\int_{\Omega} K(x, z)\, G(z - y)\, dz\Big] f(y)\, dy \tag{6}$$

où la fonction $G(z - y)$ admet une singularité forte pour $z = y$, la fonction $K(x, v)$ — la singularité faible pour $x = z$, et x est un point fixé à l'intérieur du domaine arbitraire $\Omega_1, \ldots, \Omega_m$.

Démonstration. La fonction $f(x)$ étant de classe $\mathfrak{H}_{\alpha}^{h}$, la fonction définie par l'intégrale singulière

$$\varphi(z) = \int_{\Omega} G(z - y) f(y)\, dy = \lim_{\eta \to 0} \int_{\Omega - \Pi(z, \eta)} G(z - y) f(y)\, dy \tag{7}$$

appartient aussi à la même classe $\mathfrak{H}_{\alpha}^{h}$, conformement au travail [4]. On a désigné par $\Pi(z, \eta)$ une sphère de rayon arbitraire η, centrée en z. Donc, le membre gauche de l'égalité (6) a un sens déterminé en tout point $x \in \Omega$.

Nous allons démontrer la propriété limite suivante

$$\lim_{\eta \to 0} \Big\{ \int_{\Omega} K(x, z)\Big[\int_{\Omega - \Pi'(z, \eta)} G(z - y) f(y)\, dy\Big]\, dz \Big\} = \tag{8}$$

$$= \int_{\Omega} K(x, z)\Big[\int_{\Omega} G(z - y) f(y)\, dy\Big]\, dz$$

où $\Pi'(z, \eta) = \Pi(z, \eta) \times \Omega$ est un ensemble de tous les points communs à la sphère $\Pi(z, \eta)$ et au domaine Ω.

Dans ce but il suffit de démontrer que l'intégrale itérée singulière

$$J(x,\eta)=\int_{\Omega} K(x,z)\Big[\int_{\Pi'(z,\eta)} G(z-y)f(y)\,dy\Big]dz \tag{9}$$

tend vers zéro si $\eta \to 0$. D'après l'hypothèse (2), nous pouvons écrire, dans le cas $|z-z_S|<\eta$,

$$\int_{\Pi'(z,\eta)} G(z-y)f(y)\,dy=\int_{\Pi'(z,\eta)-\Pi_S(z)} G(z-y)f(y)\,dy+\int_{\Pi_S(z)} G(z-y)\,[f(y)-f(z)]\,dy \tag{10}$$

où $\Pi_S(z)$ désigne une sphère, centrée en z, tangente au point z_S de plus courte distance des surfaces S_ν; la dernière intégrale contient une faible singularité.

En utilisant la méthode de transformation homothétique, ([4] et [5]), nous aurons alors une limitation

$$\left|\int_{\Pi'(z,\eta)} G(z-y)f(y)\,dy\right|<\text{Const}\int_0^{z\ z_S|} \frac{\varrho^{n-1}\,d\varrho}{[|z-z_S|-\varrho]^{\alpha+h}}+ \tag{11}$$

$$+\text{Const}\int_{\Pi(z,\eta)-\Pi_S(z)} \frac{dy}{|y-z|^n|\,y-y_S|^\alpha}<\frac{\text{Const}}{|z-z_S|^\alpha};\ (|z-z|<\eta)$$

indépendante de rayon η. Dans le cas $|z-z_S|\geqslant\eta$ on aura

$$\left|\int_{\Pi(z,\eta)} G(z-y)f(y)\,dy\right|<\frac{\text{Const}}{|z-z_S|^\alpha}\int_0^{\eta/|z-z_S|} \frac{ds}{s^{1-h}(1-s)^{\alpha\cdot h}}. \tag{12}$$

D'aprés les inégalités (11) et (12), à tout nombre positif ε on peut faire correspondre un ensemble de couches mesurables C_ε situées dans Ω_0, contenant à l'intérieur les surfaces $S_1, \ldots, S_p$, dont les points z sont situés suffisamment prés des surfaces correspondantes $S_0, S_1, \ldots, S_p$ pour que la partie de l'intégrale (9) étendue à la région C_ε ait une valeur absolue moindre que $\varepsilon/2$:

$$\left|\int_{C_\varepsilon} K(x,z)\Big[\int_{\Pi'(z,\eta)} G(z-y)f(y)\,dy\Big]dz\right|<\frac{\varepsilon}{2} \tag{13}$$

quel que soit η. L'ensemble C_ε étant fixé, la distance $|z-z_S|$ pour la partie extérieure $\Omega-C_\varepsilon$ admet une borne inférieure positive et nous pouvons ensuite, en vertu de (12), faire correspondre au nombre ε un rayon η_ε suffisamment petit pour qu'on ait

$$\left|\int_{\Pi(z,\eta)} G(z-y)f(y)\,dy\right|<\frac{\varepsilon}{2\varkappa},\ \text{si}\ z\in\Omega-C_\varepsilon,\ \eta<\eta_\varepsilon \tag{14}$$

où $\varkappa=\sup_{x\in\Omega}\int_{\Omega}|K(x,z)|\,dz$. En rapprochant les inégalités (13) et (14), nous aurons

$$|J(x,\eta)|<\varepsilon,\ \text{si}\ \eta<\eta_\varepsilon$$

donc $J(x, \eta \to 0$, si $\eta \to 0$ et la propriété (8) se trouve démontrée. Remarquons maintenant que l'intégrale itérée entre les accolades (8), étendue à la region $[y \in \Omega - \Pi'(z, \eta);\ z \in \Omega]$, ne contient qu'une singularité faible $z = x$ fixée à l'intérieur de Ω. Nous pouvons donc écrire l'égalité suivante

$$(15)\qquad \int_{\Omega} K(x, z)\Big[\int_{\Omega - \Pi'(z,\eta)} G(z-y) f(y)\, dy\Big] dz = \int_{\Omega}\Big[\int_{\Omega - \Pi'(y,\eta)} K(x, z)\, G(z-y)\, dz\Big] f(y)\, dy$$

où $\Pi'(y-\eta) = \Pi(y, \eta) \times \Omega$, $\Pi(y, \eta)$ désignant une sphère de rayon η, centrée en y.

Nous allons démontrer la propriété limite

$$(16)\qquad \lim_{\eta \to 0} \int_{\Omega}\Big[\int_{\Omega - \Pi'(y,\eta)} K(x, z)\, G(z-y)\, dz\Big] f(y)\, dy = \int_{\Omega}\Big[\int_{\Omega} K(x, z)\, G(z-y)\, dz\Big] f(y)\, dy.$$

L'intégrale relative à z admet une singularité forte pour $z = y$ et son existence est assurée en vertu de la transformation (voir (2) et (4))

$$(17)\qquad F(x, y) = \int_{\Omega} K(x, z)\, G(z-y)\, dz = \int_{\Pi(y,\eta)} [K(y, z) - K(x, y)]\, G(z-y)\, dz + \int_{\Omega - \Pi(y,\eta)} K(x, z)\, G(z-y)\, dz\,;\ (x \neq y;\ \Pi \subset \Omega).$$

On peut montrer, d'une façon analogue à celle de [4] pour les fonctions de classe $\mathfrak{H}_a^h$, que la fonction (17) vérifie une inégalité à singularité faible

$$(18)\qquad |F(x, y)| < \frac{\text{Const}}{|x-y|^{\gamma}\,|y-y_0|^{\sigma}};\quad (x \neq y)$$

et une condition généralisée de Hölder

$$(19)\qquad |F(x, y) - F(x, y')| < \frac{\text{Const}\ |y-y'|^{h_0}}{|x-y|^{\gamma+h_0}\,|y-y_0|^{\sigma+h_0}}$$

en supposant que $0 < \gamma < n$; $0 < h_0 \leqslant h$; $\gamma + h_0 < n$, $|x-y| \leqslant |x-y'|$. y_0 est un point de la surface S_0 où la distance $|y-y_0|$ atteint sa borne inférieure, σ étant un nombre positif arbitrairement petit.

Nous en concluons que l'existence de l'intégrale itérée (16) est assurée en tout point $x \subset \Omega$. Pour démontrer la propriété (16), il suffit de prouver que l'intégrale singulière

$$(20)\qquad I(x, \eta) = \int_{\Omega}\Big[\int_{\Pi'(y,\eta)} K(x, z)\, G(z-y)\, dz\Big] f(y)\, dy$$

tend vers zéro si $\eta \to 0$. Écrivons donc la décomposition

$$(21)\qquad \int_{\Pi'(y,\eta)} K(x, z)\, G(z-y)\, dz = \int_{\Pi_0(y)} [K(x, z) - K(x, y)]\, G(z-y)\, dz + \int_{\Pi'(y,\eta) - \Pi_0(y)} K(x, z)\, G(z-y)\, dz$$

en posant que $|y - y_0| \leqslant \eta$, $\Pi_0(y)$ désigne une sphère, centrée en y, de rayon $|y - y_0|$.

Si Π_1 désigne l'ensemble de tous les points z de la sphère $\Pi_0(y)$ pour lesquels $|z - x| \geqslant |x - y$ et Π_2 — l'ensemble de tous les points de cette sphère pour lesquels $|z - x < x - y$, on aura alors (d'aprés (2), (4), (5)) une limitation

$$(22) \qquad \left| \int_{\Pi'(y,\eta)} K(x,z)\, G(z-y)\, dz \right| < \text{Const} \int_{\Pi_1} \frac{|z-y|^{h_0}\, dz}{|x-y|^{\gamma+h} |z-y|^n} +$$

$$+ \text{Const} \int_{\Pi_2} \frac{|z-y|^{h_0}\, dz}{|z-x|^{\gamma+h_0} |z-y|^n} + \text{Const} \int_{\Pi'-\Pi_0} \frac{dz}{|x-z|^{\gamma} |z-y|^n} <$$

$$< \frac{\text{Const}}{|x-y|^{\sigma+h} |y-y_0|^{\sigma}}$$

indépendante de rayon $\eta \geqslant |y - y_0|$, $\sigma > 0$ étant arbitrairement petit; $x \in \Omega$ — fixé. Dans le cas $\eta < |y - y_0|$ on aura plus simplement, par un raisonnement analogique, une limitation

$$(22') \qquad \left| \int_{\Pi(y,\eta)} K(x,z)\, G(z-y)\, dz \right| < \frac{\text{Const}}{|x-y|^{\gamma+h_0}}$$

indépendante de rayon η. En tenant compte des inégalités (22) et (22'), nous concluons qu'à tout nombre positif ε on peut faire correspondre une sphère $\Pi_\varepsilon(x)$, centrée en x, de rayon R_ε suffisamment petit, pour qu'on ait

$$(23) \qquad \left| \int_{\Pi_\varepsilon(x)} \left[\int_{\Pi'(y,\eta)} K(x,z)\, G(z-y)\, dz \right] f(y)\, dy \right| < \frac{\varepsilon}{3}$$

quel que soit η. Ensuite, en vertu de (22), au même nombre ε on peut faire correspondre une couche $C_\varepsilon^{(0)}$ située entre la surface S_0 et la surface S_ε à l'intérieur de Ω suffisamment près de la surface S_0 pour qu'on ait

$$(24) \qquad \left| \int_{C_\varepsilon^{(0)}} \left[\int_{\Pi'(y,\eta)} K(x,z)\, G(z-y)\, dz \right] f(y)\, dy \right| < \frac{\varepsilon}{3}$$

quel que soit η. La sphère $\Pi_\varepsilon(x)$, et la couche $C_\varepsilon^{(0)}$ étant fixées, nous pouvons admettre que η est inférieur à la borne inférieure positive δ_ε de la distance $|y - y_0$ pour $y \in \Omega - C_\varepsilon^{(0)}$ et nous aurons, après avoir effectué un calcul analogue à (22), une limitation

$$(25) \qquad \left| \int_{\Pi(y,\eta)} K(x,z)\, G(z-y)\, dz \right| < \frac{\text{Const}}{R_\varepsilon^{\gamma+h_0}} \int_0^\eta \varrho^{h_0-1}\, d\varrho = \frac{\text{Const}}{R_\varepsilon^{\gamma+h_0}} \eta^{h_0}$$

si $y \in \Omega - \Pi_\varepsilon(x) - C_\varepsilon^{(0)}$, $\eta < \delta_\varepsilon$. Maintenant nous constatons qu' au nombre ε on peut faire correspondre un nombre η_ε suffisamment petit pour qu' on ait

$$(26) \qquad \left| \int\limits_{[\Omega - \Pi_\varepsilon(x) - C_\varepsilon^{(0)}]} \left[\int\limits_{\Pi'(y,\eta)} K(x,z)\, G(z-y)\, dz \right] f(y)\, dy \right| < \frac{\varepsilon}{3}$$

si $\eta < \min(\eta_\varepsilon, \delta_\varepsilon)$. En rapprochant les résultats (23), (24), (26), on arrive à la propriété (16). En vertu des propriétés démontrées (8), (15), (16), nous déduisons la thèse (6) du théorème 1.

THÉORÈME 2. *Admettant les mêmes hypothèses que dans le Théorème 1, l'égalité des intégrales itérées suivantes est vraie*

$$(27) \qquad \int\limits_\Omega G(x-z) \left[\int\limits_\Omega K(z,y) f(y)\, dy \right] dz = \int\limits_\Omega \left[\int\limits_\Omega G(x-z)\, K(z,y)\, dz \right] f(y)\, dy$$

la fonction $G(x-z)$ admettant une singularité forte pour $z = x$.

La démonstration de ce Théorème est analogique à la prédédente.

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(INSTYTUT MATEMATYCZNY, PAN)

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MATHEMATICS

On the Space of Measurable Sets of Real Numbers

by

R. ENGELKING

Presented by K. KURATOWSKI on December 10, 1960

For any two sets of real numbers A and B with finite Lebesgue measure we write $A \sim B$ if $\mu(A \dot{-} B) = 0$. It is obvious that $\sim$ is an equivalence relation. In the set $\mathfrak{M}$ of its equivalence classes we can introduce a distance ϱ, putting

$$\varrho(A_1, A_2) = \mu(A_1 \dot{-} A_2).$$

Strictly speaking, on the left hand side of the above equality we should have written a symbol for the equivalence class of sets A_1 and A_2; it seems, however, that there is no ambiguity in denoting a set and its equivalence class by the same symbol. We consider $\mathfrak{M}$ with the distance ϱ as a metric space. The space $\mathfrak{M}$ was first investigated by Fréchet and Nikodym. The enumeration of some basic properties of $\mathfrak{M}$ can be found in Halmos book [1]; in particular, $\mathfrak{M}$ is a separable, complete, convex and locally convex metric space. One can prove ([1], Theorem C and exercise (6), Sec. 41) that if we replace the set of real numbers by any measurable set in euclidean space R^n whose Lebesgue measure is infinite, we obtain a metric space $\mathfrak{M}'$ which is isometric with $\mathfrak{M}$. Therefore, without loss of generality we can restrict ourselves to considering the space of measurable subsets of real numbers. $\mathfrak{M}$ is a metric group with symmetric difference as a group operation.

Translations in $\mathfrak{M}$ are isometries, hence it is a metrically homogeneous space.

An investigation of the degree of homogencity of $\mathfrak{M}$ led us to the following.

THEOREM 1. *For any two systems A_1, A_2, A_3 and B_1, B_2, B_3 of elements of $\mathfrak{M}$, such that $\varrho(A_i, A_j) = \varrho(B_i, B_j)$ $(i, j = 1, 2, 3)$, there exists an isometry f of $\mathfrak{M}$ onto itself such that $f(A_i) = B_i$ $(i = 1, 2, 3)$.*

An example can also be constructed showing that the number 3 in Theorem 1 cannot be replaced by a greater number.

The degree of universality of $\mathfrak{M}$ is described by the following

THEOREM 2. *Every metric space consisting of four points is isometric with a subset of $\mathfrak{M}$. There exists a five-points metric space which can not be isometrically embedded into $\mathfrak{M}$.*

The above Theorems are of metric character. Topological counterparts of the Theorems are

THEOREM 3. *For any two systems* $A_1, A_2, \ldots, A_n$ *and* $B_1, B_2, \ldots, B_n$ *of different elements of* $\mathfrak{M}$, *there exists a homeomorphism* f *of* $\mathfrak{M}$ *onto itself such that*

$$f(A_i) = B_i \quad (i = 1, 2, \ldots, n).$$

and the following simple and certainly known but unpublished to the best knowledge of the author.

THEOREM 4. $\mathfrak{M}$ *contains topologically the Hilbert cube, and therefore* $\mathfrak{M}$ *is a topologically universal space for metric separable spaces.*

Proofs of the above theorems will be published in Prace Matematyczne.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY, PAN)

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MATHEMATICS

A Metrical Characterization of n-Cells

by

W. NITKA

Presented by K. BORSUK on December 17, 1960

Let $\langle X, \varrho \rangle$ be a metric space with distance function ϱ. A point $q \in X$ is said to *lie between* p *and* r, $p, r \in X$ (writing pqr), provided that

$$\varrho(p, r) = \varrho(p, q) + \varrho(q, r).$$

We say that the set $A \subset X$ is *linear*, if there exists an isometrical transformation $i: A \to E^1$ of A into the set E^1 of real numbers, i.e. $\varrho(p, q) = |i(p) - i(q)|$, for every $p, q \in A$.

Evidently, if pqr, then the set $\{p, q, r\}$ is linear.

A topological arc $L \subset X$ is called a *metric segment*, if L is linear.

A space $\langle X, \varrho \rangle$ is said to be *convex* ([5], p. 81), provided that for each two distinct points $p, r \in X$ there exists a point $q \in X$ such that $p \neq q \neq r$ and pqr.

It is well known (see, for ex. [1], p. 41) that in complete convex spaces X each two points $p, q \in X$ are joined by a metric segment.

A metric space $\langle X, \varrho \rangle$ is said to be *strongly convex* (SC-space), if it is: 1° complete, 2° convex and 3° the following condition (α) holds:

(α) if prq and psq, then the set $\{p, q, r, s\}$ is linear.

This notion was studied also by K. Borsuk [2]. It is not difficult to prove that, if $\langle X, \varrho \rangle$ is SC-space, then each two distinct points $p, q \in X$ are joined by exactly one metric segment, denoted by $\overline{pq}$.

A metric space $\langle X, \varrho \rangle$ is said to be *without ramifications* (WR-space), if it is 1° complete, 2° convex and 3° the following condition (β) holds:

(β) if pqr, pqs and $p \neq q$, then the set $\{p, q, r, s\}$ is linear.

This notion was studied also by H. Buseman ([3] p. 36).

A subset S of a SC-space $\langle X, \varrho \rangle$ is called *convex*, if for every two points p, q of S we have $\overline{pq} \subset S$.

A convex metric space $\langle X, \varrho \rangle$ is said to have the *convex triangle property* (CT-space), if the following metric condition holds:

(CT) for every three points $v, p, q \in X$ the set $\bigcup_{r \in \overline{pq}} \overline{vr}$ is convex.

Let us consider the following condition (a kind of the Pasch-axiom) in SC-spaces:

(*P*) for every six points $u, v, w, u', v', w' \in X$ such that $vu'w$, $wv'u$ and $uw'v$, there exists a point z such that vzv' and $u'zw'$

THEOREM 1. *In compact* SC-WR-*space the conditions* (CT) *and* (*P*) *are equivalent.*

The following characterization theorem solves a problem of K. Borsuk:

THEOREM 2. *If* $\langle X, \varrho \rangle$ *is a compact* SC-WR-CT-*space and dim* $X = n$, *then* X *is an* n-cell.

On account of Theorem 1 the condition CT in Theorem 2 can be replaced by the condition (*P*). Obviously, since the cartesian metric of the cube I^n is SC-WR-CT, Theorem 2 gives a characterization of n-cells.

It was proved [4] that, if $\langle X, \varrho \rangle$ is a compact SC-WR-space and dim $X = 2$, then X is a 2-cell. This result becomes a special case of main Theorem 2 since the following implication holds:

THEOREM 3. *If* $\langle X. \varrho \rangle$ *is a compact* SC-WR-*space and dim* $X \leqslant 2$, *then* X *is also a* CT-*space.*

The complete proofs will be published in the paper On convex metric spaces II, in Fundamenta Mathematicae.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY, PAN)

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MATHEMATICS

On a New Definition of Polynomials

by

R. BITTNER

Presented by S. MAZUR on December 23, 1960

1. Introduction

The well-known definition of a polynomial is as follows: a polynomial is an expression of the form

$$a_0 + a_1 t + \ldots + a_n t^n,$$

where t is an indeterminate, and $a_0, a_1, \ldots, a_n$ are elements of a commutative ring R. Writing $a_0 + a_1 t + \ldots + a_n t^n = \sum_{k=0}^{\infty} a_k t^k$ (where $a_k = 0$ for $k > n$) we define addition and multiplication of polynomials

$$\sum_{k=0}^{\infty} a_k t^k + \sum_{k=0}^{\infty} b_k t^k = \sum_{k=0}^{\infty} (a_k + b_k) t^k,$$

$$\left(\sum_{k=0}^{\infty} a_k t^k \right) \cdot \left(\sum_{k=0}^{\infty} b_k t^k \right) = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} a_i b_{k-i} \right) t^k$$

so that we obtain a commutative ring $R[t]$ containing R. We can also consider in the rings with characteristic ∞ a formal derivative S of a polynomial defined by the formula

$$S\left(\sum_{k=0}^{\infty} a_k t^k \right) = \sum_{k=1}^{\infty} k a_k t^{k-1}.$$

This derivative is a homomorphism of the ring $R[t]$. The formula

$$S(x \cdot y) = Sx \cdot y + x \cdot Sy$$

holds for every pair of polynomials x, y.

This paper treats an inverse problem in linear space C^0 over a field Γ with characteristic ∞. Let us consider a linear operation S, called derivative, of a linear space C^1 into the linear space C^0, $C^1 \subset C^0$. Let $K(S)$ be the set of all constants a defined by the formula $Sa = 0$ and let $K^{\infty}(S)$ be the set of all polynomials x defined

by the formula $S^n x = 0$ for some integer n. If the set $K(S)$ forms an algebra under addition $+$, multiplication by scalars and multiplication $a \cdot b$ with unity l, then multiplication for polynomials $x, y \in K^\infty(S)$ can be defined such that $K^\infty(S) = K(S)[t]$ and the linear operation S is the derivative in the ring $K^\infty(S)$.

This theorem and its corollaries give some information about problems in papers [3] and [4].

2. The ring of polynomials

Let C^n be the domain of S^n. If there exist some integers n such that $x \in C^n$ and $S^n x = 0$ then x is called a polynomial. We say that a polynomial x has the degree n if $S^{n+1} x = 0$ and $S^n x = 0$.

Let T be a given endomorphism of C^0 such that $STx = x$ for $x \in C^0$. There are many such endomorphisms called integrals, if $K(S)$ is not trivial. For given S and T the operation $sx = x - TSx$, called limit condition, is the linear operation from C^1 into $K(S)$.

THEOREM 1. *Every polynomial x of degree n has the form*

$$x = a_0 + T a_1 + \ldots + T^n a_n, \tag{1}$$

where $a_i = sS^i x$, $i = 0, 1, \ldots, n$ are constants uniquely determined by element x and endomorphism T.

Let us suppose that the set $K(S)$ forms a commutative algebra over the field Γ with operations $+$ and $\cdot$ and with the unit of multiplication l.

We define multiplication $x \cdot y$ of polynomials $x = \sum_{k=0}^{\infty} T^k a_k$, $y = \sum_{k=0}^{\infty} T^k b_k$ (where $a_k = 0$, $b_k = 0$ for great k) by the formula

$$\left(\sum_{k=0}^{\infty} T^k a_k \right) \cdot \left(\sum_{k=0}^{\infty} T^k b_k \right) = \sum_{k=0}^{\infty} T^k \left(\sum_{k=0}^{\infty} \binom{k}{i} a_i b_{ki} \right). \tag{2}$$

THEOREM 2. *The set $K^\infty(S)$ of all polynomials forms under addition $+$ and multiplication $\cdot$ a commutative algebra over the field Γ.*

If we denote Tl by t then $n!\, T^n l = t^n$ for $n = 0, 1, \ldots$, so that

$$\sum_{k=0}^{\infty} T^k a_k = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}. \tag{3}$$

THEOREM 3.

$$S(x \cdot y) = Sx \cdot y + x \cdot Sy. \tag{4}$$

Let T be another endomorphism of C^0 such that $\bar{S}\bar{T}x = x$ for $x \in C^0$. We can define another multiplication $x \cdot y$ of polynomials $x = \sum_{k=0}^{\infty} \bar{T}^k a_k$, $y = \sum_{k=0}^{\infty} T^k b_k$ by the formula

$$\left(\sum_{k=0}^{\infty} T^k a_k \right) \cdot \left(\sum_{k=0}^{\infty} T^k b_k \right) = \sum_{k=0}^{\infty} T^k \left[\sum_{i=0}^{k} \binom{k}{i} a_i b_{k-i} \right]. \tag{5}$$

Let be $t = Tl$, $t^k = \underbrace{t \cdot t \cdot \ldots t}_{k \text{ times}}$. Then we have

THEOREM 4. (Newton's formula)*)

$$\bar{t}^n \alpha = \sum_{k=0}^{n} \binom{n}{k} t^{n-k} s(\bar{t}^k \alpha) \quad \text{for } \alpha \in K(S). \tag{6}$$

THEOREM 5. *The multiplication $x \cdot y$ does not depend on the choice of endomorphism T if and only if*

$$s(t^k) = [st]^k \quad \text{for } k = 2, 3, \ldots . \tag{7}$$

Example A. The derivative $S\{a_n\} = \{a_{n+1} - a_n\}$ in the space of sequences $\{a_n\}$, $n = 0, 1, \ldots$ of real numbers has integrals

$$T_k\{a_n\} = \{\underbrace{-a_0 - a_1 - \ldots - a_{k-1}}_{0}, \underbrace{-a_1 - \ldots - a_{k-1}}_{1}, \ldots, \underbrace{-a_{k-1}}_{k-1}, \underbrace{0}_{k}, \underbrace{a_k}_{k+1}, \underbrace{a_k + a_{k+1}}_{k+2}, \ldots\}$$

for $k = 0, 1, \ldots$ In particular

$$T_1\{a_n\} = \{-a_0, 0, a_1, a_1 + a_2, \ldots\}.$$

The sequences $a = \{a\}$ are constants and $l = \{l\}$ is the unit of the ring of constants with the usual multiplication. We have for $t = T_1 l$

$$s_{T_0}(t^2) = \{2\} \neq [s_{T_0} \bar{t}]^2 = \{1\}.$$

We see that the derivative S has many different multiplications, $x \cdot y$ satisfying (4).

3. Divisibility in algebras

Given the integral T for every other integral $\bar{T}$, and $t = Tl$, $t = Tl$ we have

$$x = f(t) = f(\bar{t}).$$

We say that for $a = Tl - \bar{T}l = t - t \in K(S)$ the polynomial x has the value

$$\bar{f}(a) = s_{\bar{T}} x.$$

THEOREM 6. *If $\bar{\bar{\alpha}} = \bar{\alpha} - a$ and $\bar{f}(\bar{\alpha})$ exist, then $f(\bar{\alpha})$ exists and*

$$f(\alpha) = f(\bar{\alpha}) \tag{8}$$

so that, if $\bar{f}(t) = f(t)$, then $\bar{f}(\alpha) = f(\bar{\alpha})$.

We see that the counterdomain of the polynomial does not depend on translation. If follows that, if we can give the values of a polynomial by given integral T, then

*) If $S\{x(t)\} = \{x'(t)\}$, $T\{x(t)\} = \left\{\int_0^t x(\tau)\, d\tau\right\}$, $\bar{T}\{x(t)\} = \left\{\int_{-b}^{t} x(\tau)\, d\tau\right\}$, $l = \{1\}$, then formula (6) gives

$$(t + b)^n = \sum_{k=0}^{n} \binom{n}{k} t^{n-k} b^k$$

and we have Newton's formula by $t = a$.

we have all values by other integrals. Taking an integral T we introduce a co-ordinate system in the set $K(S)$ of constants. Then for $x = f(t) - f(\bar{t})$, where $t = Tl$, $\bar{t} = \bar{T}l$, $a = t - \bar{t} = s_{\bar{T}} Tl$, we have

$$s_T x = f(0), \quad s_{\bar{T}} x = f(a) = f(0).$$

We say that a polynomial x is divisible in $\dot{-}$ way by a polynomial y, if there is some polynomial z such that $x = y \dot{-} z$. If this is the case, we write $y \mid x\,(\dot{-})$.

THEOREM 7. The Division Algorithm. *If x is any polynomial and if $y = t^n + a_{n-1} t^{n-1} + \ldots + a_0 \in K^\infty(S)$, then we can find polynomial $q, r \in K^\infty(S)$ such that*

$$x = q \cdot y + r, \tag{9}$$

where r is either zero or of a degree lower than n. The representation (9) *is unique.*

COROLLARY 1. *The remainder r of a polynomial $x = W(t) - \overline{W}(t)$ when divided in $\dot{-}$ way by $t = t - a$ is $W(a) = \overline{W}(0)$* (Remainder theorem) *so that*

$$\bar{t} \mid \overline{W}(t)\,(\dot{-}) \text{ if and only } W(a) = 0. \tag{10}$$

COROLLARY 2. *For every pair of polynomials $x, y \in K^\infty(S)$ and for $d =$ g. c. d. (x, y) there exist polynomials a, b such that*

$$d = a \cdot x + b \cdot y.$$

4. Polymomials of many variables and distributive polynomials

We can consider in general n derivatives $S_1, \ldots, S_n$ and n integrals $T_1, \ldots, T_n$ such that

$$S_i T_i = I, \quad S_i S_j = S_j S_i, \quad T_i T_j = T_j T_i \quad \text{for } i, j = 1, \ldots, n.$$

We have then n limit conditions $s_1, \ldots, s_n$, where

$$s_i = I - T_i S_i \text{ for } i = 1, \ldots, n.$$

We call x the polynomial of many variables of the degree $[m_1, \ldots, m_n]$, if for the variable $t_i = T_i l$ it is a polynomial of degree m_i. Every polynomial of many variables has the unique form

$$x = \sum_{0 \leq i_1, \ldots, 0 \leq i_n} T_1^{i_1} \ldots T_n^{i_n} a_{i_1 \ldots i_n},$$

where $a_{i_1 \ldots i_n} \in \bigcap_{i\ 0}^{n} K(S_i)$ and $a_{i_1 \ldots i_n} = 0$ for $i_1 > m_1, \ldots, i_n > m_n$.

We can define multiplication and values of polynomials of many variables similarly as in points 2—3 and similar theorems can be obtained without the Division Algorithm and Corollary 2.

We can consider also distributive polynomials, that is elements, $x \in K^\sim(S_1 \ldots S_n)$. We then take a new derivative

$$S = S_1 \ldots S_n,$$

called distributional derivative. This gives us the distributional integral

$$T = T_1 \ldots T_n$$

and the distributional limit condition

$$s = I - TS = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{i_\mu + i_\nu \\ (i_1, \ldots, i_k)}} s_{i_1} s_{i_2} \ldots s_{i_k} x .$$

The constants are the elements a for which $Sa = 0$. These elements have the form $a = a_1 + a_2 + \ldots + a_n$, where $a_i \in K(S_i)$. Element x is the distributive polynomial, if $S^m x = 0$ for some integer m. We see that the definition of distributive polynomials is more general than the definition of polynomials of many variables. Multiplication and values of distributive polynomials are defined as above and all Theorems 1—7 can be obtained.

Example B. We have seen that the difference operation $S\{a_n\} = \{a_{n+1} - a_n\}$ can be taken as a derivative S. One of its multiplications has the form

$$\{a_n\} \cdot \{b_n\} = \Big\{ \sum_{\sigma + \varrho + \tau \quad n} (-1)^{\sigma} a_\sigma b_\tau \Big\}.$$

Let L be a differential operation, that is

$$L\{u(x, y)\} = \sum_{i,j=0}^{2} a_{ij}(x, y) \frac{\partial^2 u}{\partial x_i \partial x_j},$$

where $a_{ij}(x, y)$ are continuous functions over the domain Ω and let $Vu = 0$ be a limit condition on the boundary $\partial\Omega$ of the considered domain Ω. Then a function su such that $Lsu = 0$, $V(su - u) = 0$ is the limit condition in the sense considered above, if for derivative L there exists a uniquely determined integral T satisfying conditions $LT = I$, $VT = 0$.

We can multiply constants multiplying their values on the boundary. Multiplying constants we multiply polynomials by formula [2]. We see that the heat, wave, harmonic and other differential equations introduce their polynomials as well as difference, differential-difference and other equations. All this can be done under one essential condition: the integral T is an endomorphism of the considered space C^0 and there are no trivial constants.

Example C. If we take $S = \dfrac{\partial^2 u}{\partial x \, \partial y}$, the simplest limit condition is $s\{(x, y)\} = \{u(x, 0) + u(0, y) - u(0, 0)\}$ for integrals $T_1 = \int_0^x$, $T_2 = \int_0^y$. But we can also obtain the Szmydt condition [6] taking $T_1 = \int_{\varphi(y)}^{x}$, $T_2 = \int_{\psi(x)}^{y}$, where $x = \varphi(y)$ and $y = \psi(x)$ are two continuous curves in the domain Ω of points (x, y), and (x_0, y_0) is their common point. We have then the limit condition

$$\bar{s}\{u(x, y)\} = \{u(\varphi(y), y) + u(x, \psi(x) - u(x_0, y_0)\}$$

similarly as in the Szmydt condition, where the values $u(\varphi)(y), y)$ and $u(x, \psi(x))$ must be known. It is well known that the Szmydt condition gives for the hyperbolic equation the Cauchy, Darboux and Picard conditions for special curves $x = \varphi(y)$ and $y = \psi(x)$.

We multiply constants by the formula

$$\{u(x,0)+u(0,y)-u(0,0)\}\cdot\{v(x,0)+v(0,y)-v(0,0)\}=$$
$$=\{u(x,0)\,v(x,0)+u(0,y)\,v(0,y)-u(0,0)\,v(0,0)\}.$$

By this multiplication functions $\{x\}$ and $\{y\}$ are not relatively prime because

$$\{x\}=\left\{\frac{1}{1+x^2}+\frac{1}{1+y^2}-1\right\}\{x(1+x^2)\},$$

$$\{y\}=\left\{\frac{1}{1+x^2}+\frac{1}{1+y^2}-1\right\}\{y(1+y^2)\}.$$

The distributive polynomials of the degree m are such functions continuously derivable with their derivatives that $\dfrac{\partial m}{\partial y^m}$ is a polynomial of x of m-th degree with coefficients depending of x.

Considering for the heat equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$$

the boundary problem

$$u(x,0)=\varphi(x)$$
$$u(0,t)=\Theta_1(t),\quad u(l,t)=\Theta_2(t),$$

we can take the distributional derivative $S=\dfrac{\partial^3}{\partial t\,\partial x^2}$, the integrals $T_1\cdot\int_0^t$, $T_2\{u(x,t)\}=\left\{\int_0^1 G(x,\xi)\,u(\xi,t)\,d\xi\right\}$, where G is the Green's function of operator $\dfrac{\partial^2}{\partial x^2}$, and the limit condition

$$s\{u(x,t)\}=\left\{u(x,0)\quad\frac{l-x}{l}[u(0,t)-u(0,0)]+\frac{x}{l}[u(l,t)-u(l,0)]\right\}.$$

Distributive polynomials u of degree k are such functions u that

$$\frac{\partial^{3k}u}{\partial t^k\,\partial x^{2k}}=0.$$

NORMAL SCHOOL, GDAŃSK
(WYŻSZA SZKOŁA PEDAGOGICZNA, GDAŃSK)

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Evaluation of the Phase Space Integral in the Statistical Treatment of Multiple Production with Angular Momentum Conservation

by

R. RĄCZKA

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Introduction

In the usual formalism of Fermi's statistical theory [1] of multiple particle production it is very difficult to take into account the conservation of the angular momentum except by a crude thermodynamical approximation [2]. On the other hand, the experimental data strongly suggest that a transfer of a large angular momentum plays an essential role in many physical events.

In order to introduce the conservation of the angular momentum into the statistical theory, Koba [3] proposed recently to replace the momentum representation by a representation in which angular momentum L, one of its components m and the magnitude of linear momentum (p) are good quantum numbers. His final expression for the relative probability of transition into an N particle final state, with total energy E and total angular momentum L, can be written as

$$(1)\quad S_N = A\left(\frac{2}{\pi}\right)^N \sum_{l_1,\ldots,l_N}^{(L)} \frac{Z(L;l_i)}{\Pi(n_\nu)!} \cdot \prod_{i=1}^{N} \left\{ \mu_i \int_0^R r_i^2\, dr_i \int_0^\infty \frac{p_i^2}{E_i} j_{l_i}^2 (r_i p_i)\, dp_i \right\} \frac{dH(E - \sum_i^N E_i)}{dE}.$$

In this expression $A(\tau, N)$ denotes the dependence of isotopic spins. n_ν is the number of particles of the same kind and with indentical angular momentum, and $Z(L_i l_1, \ldots, l)$ is the multiplicity of the states with angular momentum L involved in the product of the individual states with angular momenta $l_1, \ldots, l_n$. $H(x)$ is Heaviside's step function.

In order to apply this new formulation to concrete physical problems, it is necessary to work out the integration in the formula for S_N with a reasonable approximation.

In this paper we shall perform the calculation of this phase space integral in two physically interesting cases. The first case is concerned, roughly speaking, with larger energy of the colliding particles and the second one with smaller energy.

Calculation of the phase space integrals

Using the Fourier transform for Heaviside's function

$$H(x) = \frac{1}{2\pi i} \int_{-\infty-i\varepsilon}^{+\infty-i\varepsilon} \alpha^{-1} e^{i\alpha x} d\alpha , \tag{2}$$

and the definition of

$$j_l = \left(\frac{\pi}{2\varrho}\right)^{1/2} I_{l+\frac{1}{2}}(\varrho), \tag{3}$$

we obtain

$$S_N = \frac{A}{2\pi} \sum_{l_s}^{L} \frac{Z(L; l_i)}{\Pi (n_\nu!)} \int_{-\infty-i\varepsilon}^{+\infty-i\varepsilon} e^{iE\alpha} d\alpha \prod_{i=1}^{N} \left\{ \mu_i \int_0^R r_i dr \int_0^\infty \frac{p_i}{E_i} I^2_{l_i+\frac{1}{2}} (p_i r_i) e^{-i\alpha E} dp_i \right\}. \tag{4}$$

As calculation of these integrals for $E_i = \sqrt{p_i^2 + \mu_i^2}$ is very difficult, we restrict ourselves to extremely relativistic particles, where $E_i = p_i$, and to nonrelativistic particles, where $E_i = \mu_i + p_i^2/2\mu_i$.

In the first case we have the following type of integrals

$$\mathrm{I}_{rel} = \mu \int_0^R r dr \int_0^\infty I^2_{l+\frac{1}{2}} (rp) e^{-i\alpha p} dp . \tag{5}$$

As the integration with respect to r is over a finite domain it is convenient to use generalized hypergeometric functions instead of the Bessel functions [4]

$$I^2_{l+\frac{1}{2}}(z) = \left(\frac{z}{2}\right)^{2l+1} \Gamma^{-2}(l+\tfrac{3}{2})\ {}_1F_2 \left(\begin{matrix} l+1; & -z^2 \\ 2l+2, & l+\frac{3}{2}; \end{matrix} \right). \tag{6}$$

After some calculations we obtain:

$$\mathrm{I}^{(i)}_{rel} = \frac{\mu_i}{2\sqrt{\pi}} \cdot R^{2l_i+3} \cdot (i\alpha)^{-2l_i-2} \cdot {}_2\bar{F}^{(i)}_2 \left(\begin{matrix} 2l_i+2, & l_i+1; & 2, & 1; \\ 2l_i+2, & l_i+\frac{5}{2}; & 1, & 1; \end{matrix} \; R^2/\alpha^2 \right). \tag{7}$$

Here ${}_pF_q(z)$ is a still more general function than the generalized hypergeometric function and is defined as follows:

$${}_p\bar{F}_q(z) = {}_p\bar{F}_q \left(\begin{matrix} a_r; & c_r; \\ b_r; & d_r; \end{matrix} \; z \right) = \sum_{k=0}^{\infty} \frac{\prod_{r=1}^{p} \Gamma(a_r + c_r k)\, z^k}{\prod_{r=1}^{q} \Gamma(b_r + d_r k)\, k!} . \tag{8}$$

In the nonrelativistic case we obtain the following type of integrals

$$\mathrm{I}_{nr} = \mu e^{-i\alpha\mu} \int_0^R r dr \int_0^\infty \frac{p}{\mu + p^2/2\mu} I^2_{l+\frac{1}{2}} (rp) e^{-i\alpha p^2/2\mu} dp. \tag{9}$$

In this case we can replace $\mu+p^2/2\mu$ by μ in the denominator. Using relation (6) and putting $p^2 = x$ we obtain the Laplace transform of the hypergeometric function ${}_1F_2(-r^2 x)$. After all integrations, we finally obtain

$$(10)\qquad \mathrm{I}_{nr}^{(i)} = \left(\frac{\mu}{2}\right)^{l_i+\frac{3}{2}} R^{2l_i+3}\cdot(i\alpha)^{-l_i-\frac{3}{2}}\cdot e^{-t\alpha\mu_i}\,{}_1\bar{F}_2^{(i)}\left(\begin{matrix}2l_i+2;\,2;\,-\frac{1}{2}\mu R^2/i\alpha\\ 2l_i+2,\,l_i+\frac{5}{2};\,1,1;\end{matrix}\right).$$

Assuming that in the collision N_1 nonrelativistic particles and N_2 extremely relativistic ones are produced, we get for the relative probability of transition the following expression:

$$(11)\qquad S_{N_1+N_2} = \frac{1}{2\pi}\sum_{l_i}^{(L)}\gamma\int_{-\infty-i\varepsilon}^{+\infty-i\varepsilon} d\alpha\prod_{i=1}^{N_1}\left\{{}_1\bar{F}_2^{(i)}\left(\frac{-\mu R^2}{2i\alpha}\right)\right\}\cdot\prod_{s=1}^{N_2}\left\{{}_2\bar{F}_2^{(s)}\left(\frac{R^2}{\alpha^2}\right)\right\}\cdot$$

$$\cdot(i\alpha)^{-\varkappa}\cdot e^{i\alpha\left(E-\sum_{k=1}^{N_1}\mu_k\right)}$$

where

$$(12)\qquad \gamma = A\,\frac{Z}{\Pi\,(n_\nu!)}\cdot R^{\sum_{i=1}^{N_1+N_2}(2l_i+3)}\cdot\prod_{i=1}^{N_1}\left(\frac{\mu_i}{2}\right)^{l_i+\frac{3}{2}}\cdot\prod_{s=1}^{N_1}\left(\frac{\mu_s}{2\sqrt{\pi}}\right)$$

and

$$\varkappa = \sum_{i=1}^{N_1}\left(l_i+\frac{3}{2}\right)+\sum_{i=1}^{N_2}(2l_i+2).$$

In order to perform integration with respect to α we expand each hypergeometric function into series and we have

$$(13)\qquad S_{N_1+N_2} = \frac{1}{2\pi}\sum_{l_i}^{(L)}\gamma\prod_{i=1}^{N_1}\left\{\sum_{k_i=0}^{\infty}f_1(k_i)\right\}\cdot\prod_{s=1}^{N_2}\left\{\sum_{h_s=0}^{\infty}f_2(h_s)\right\}\cdot$$

$$\cdot\int_{-\infty-i\varepsilon}^{+\infty-i\varepsilon} e^{i\alpha\left(E-\sum^{N_1}\mu_i\right)}\cdot(i\alpha)^{-r_{N_1,N_2}}d\alpha$$

where

$$(14)\qquad \begin{cases} f_1(k) = \dfrac{\Gamma(2l+2+2k)\,[-\frac{1}{2}\mu R^2]^k}{\Gamma(2l+2+k)\,\Gamma(l+\frac{5}{2}+k)\,k!}\\[2ex] f_2(h) = \dfrac{\Gamma(2l+2+2k)\,\Gamma(l+1+k)\,(-R^2)^k}{\Gamma(2l+2+k)\,\Gamma(l+\frac{5}{2}+k)k!}\end{cases}$$

$$r_{N_1,N_2} = \varkappa+\sum_{i=1}^{N_1}k_i+\sum_{i=1}^{N_2}2k_i.$$

The last integral in the right hand side of (13) gives

$$(15)\qquad \int\limits_{-\infty-i\varepsilon}^{+\infty-i\varepsilon} e^{i\alpha}\,\alpha^{\left(E-\sum\limits^{N_1}\mu_i\right)}\cdot(i\alpha)^{-r_{N_1,N_2}}\,d\alpha = 2\pi\left(E-\sum_{i=1}^{N_1}\mu_k\right)^{r_{N_1,N_2}-1}\Big/\,\Gamma(r_{N_1,N_2}),$$

thus we get the final formula for the relative probability of transition:

$$(16)\qquad S_{N_1,N_2} = \sum_{l_s}^{(L)}\gamma\left(E-\sum^{N_1}\mu_s\right)^{\varkappa-1}\cdot\prod_{i=1}^{N_1}\left\{\sum_{k_i=0}^{\infty}f_1(k_i)\cdot\left(E-\sum^{N_2}\mu_s\right)^{k_i}\right\}\cdot$$

$$\cdot\prod_{i=1}^{N_2-1}\left\{\sum_{h_i=0}^{\infty}f_2(h_i)\left(E-\sum^{N_1}\mu_s\right)^{2h_i}\right\}\cdot{}_2\bar{F}_3^{(N_2)},$$

where

$$(17)\qquad {}_2\bar{F}_3^{(N_2)} = {}_2\bar{F}_3\left(\begin{matrix}2l_{N_2}+2,\ l_{N_2}+1;\ 2,\ 1;\\ 2l_{N_2}+2,\ l_{N_1}+\tfrac{5}{2},\ r_{N_1,N_2-1};\ 1,\ 1,\ 2;\end{matrix}\ -R^2\left(E-\sum^{N_1}\mu_s\right)^2\right).$$

The formula (16) is quite general, but to obtain the expression in a closed form we will use the asymptotic expansion ${}_p\bar{F}_q(-z)$. We shall investigate the following two cases

a) Case of large argument. *) Let us consider now the function

$$(18)\qquad {}_2\bar{F}_3^{(N_2)}\left[-R^2\left(E-\sum_{s=1}^{N_1}\mu_s\right)^2\right].$$

For large values of $E-\sum\limits_{s=1}^{N_1}\mu_s$ we can use an asymptotic expansion for the generalized hypergeometric function given by Wright [5]. According to this expansion, we have

$$(19)\qquad {}_2\bar{F}_3^{(N_2)} = \left\{2\ln\left[R\left(E-\sum_{s=1}^{N_1}\mu_s\right)\right]-1{,}76715-2\psi(l_{N_2}+1)-\right.$$

$$\left.+\,2\psi(\varkappa-2l_{N_2}-2)\right\}\cdot\left[R\left(E-\sum_{s=1}^{N_1}\mu_s\right)\right]^{-2l_{N_2}-2}:\sqrt{\pi}\,\Gamma(r_{N_1,N_2-1}-2l_{N_2}-2).$$

If in (16) we replace ${}_p\bar{F}_q^{(N_2)}$ by (19) we obtain, owing to $\Gamma^{-1}(r_{N_1,N_2-1}-2l_{N_2}-2)$ once more ${}_2\bar{F}_3^{(N_2-1)}\left[-R^2\left(E-\sum\limits^{N_1}\mu_s\right)^2\right]$, instead of the last two factors of (16). If we apply N_2 times asymptotic expansion for ${}_2\bar{F}_3^{(i)}$ $(i=1,2,\ldots,N_2)$ we get

*) This case usually corresponds to collisions in which secondary particles with large energy are created. We assume that in this case $E-\sum\limits_{s=1}^{N_1}\mu_s \geqslant 100\,\mu_\pi$, although in special cases the final result (23) is valid also for smaller values of $E-\sum\limits_{s=1}^{N_1}\mu_s$.

$$(20)\qquad S_{N_1+N_2} = \sum_{l_s}^{(L)} \gamma\left(E - \sum^{N_1} \mu_s\right)^{\varkappa'-1} \prod_{s=1}^{N_2} C_s \cdot \prod_{i=1}^{N_1-1} \left\{ \sum_{k_i=0}^{\infty} f_1(k_i) \cdot \right.$$

$$\left. \cdot \left(E - \sum^{N_1} \mu_s\right)^{k_i} \right\} \cdot {}_1\bar{F}_3^{(N_1)}\left[-\frac{1}{2}\mu R^2 \left(E - \sum^{N_1} \mu_s\right)\right],$$

where

$$(21)\qquad \begin{cases} C_s = \dfrac{1}{\sqrt{\pi}} \left\{ 2\ln\left[R\left(E - \sum_{i=1}^{N_1} \mu_i\right)\right] - 1{,}76715 - 2\psi(l_s+1) - \right. \\ \qquad \left. - 2\psi\left(\varkappa - \sum_{i=s}^{N_2}\right)(2l_i+2)\right\} \cdot R^{-2l_{N_s}-2}; \quad \varkappa' = \sum_{i=1}^{N_1}\left(l_i + \frac{3}{2}\right) \\ {}_1\bar{F}_3^{(N_1)} = {}_1\bar{F}_3\begin{pmatrix} 2l_{N_1}+2;\ 2;\ -\frac{1}{2}\mu R^2\left(E - \sum^{N_1}\mu_s\right) \\ 2l_{N_1}+2;\ l_{N_1}+\frac{5}{2},\ r_{N_1-1,0};\ 1,\ 1,\ 1; \end{pmatrix}. \end{cases}$$

Using Wright's asymptotic expansion also for the function ${}_1\bar{F}_3^{(N_1)} = [\frac{1}{2}\mu R^2(E - \sum^{N_1}\mu_s)]$ we have

$$(22)\qquad {}_1\bar{F}_3^{(N_1)} = \frac{\left[\frac{1}{2}\mu R^2(E - \sum^{N_1}\mu_s)\right]^{-l_{N_1}-1}}{\Gamma(\frac{3}{2})\cdot\Gamma(r_{N_1-1,0} - l_{N_1} - 1)} + 0\left\{\left[\frac{1}{2}\mu R^2\left(E - \sum_{s=1}^{N_1}\mu_s\right)\right]^{-2l_{N_1}-3}\right\}.$$

If we apply this expansion N_1 times, we obtain the final formula *)

$$(23)\qquad S_{N_1+N_2} = \sum_{l_i}^{(L)} \alpha\left(E - \sum_{k=1}^{N_1}\mu_k\right)^{\frac{N_1}{2}-1} \cdot \prod_{i=1}^{N_2}\gamma_i$$

where

$$(24)\qquad \alpha = A\frac{Z}{\Pi(n_\nu!)} \cdot \frac{R^{N_1+N_2}}{\Gamma(N_{1/2})} \cdot \prod_{i=1}^{N_1}\left(\frac{2\mu_i}{\pi}\right)^{1/2} \prod_{s=1}^{N_2}\frac{\mu_i}{2\pi}$$

$$\gamma_i = \left\{2\ln\left[R\left(E - \sum_{s=1}^{N_1}\mu_s\right)\right] - 1{,}76715 - 2\psi(l_i+1) - 2\psi\left[\varkappa - \sum_{s=i}^{N_2}(2l_s+2)\right]\right\}.$$

b) Case of small argument. **) For small values of the argument it is convenient to change the order of summation of generalized hypergeometric functions in general formula.

$$(25)\qquad S_{N_1+N_2} = \sum_{l_i}^{(L)} \gamma\left(E - \sum^{N_1}\mu_s\right)^{\varkappa} \cdot \prod_{i=1}^{N_2}\left\{\sum_{h_i=0}^{\infty} f_2(h_i) \cdot \right.$$

$$\left. \cdot \left(E - \sum^{N_1}\mu_s\right)^{2h_i}\right\} \cdot \prod_{s=1}^{N_1-1}\left\{f_1(k_s)\left(E - \sum^{N_1}\mu_j\right)^{k_s}\right\} \cdot {}_1\bar{F}_3^{(N_1)}.$$

*) If we have only extremely relativistic particles the last functions are ${}_2F_3^{(1)}(-z) = {}_1\bar{F}_2^{(1)}(-z)$ and its asymptotic expansion are ${}_1\bar{F}_2^{(1)}(-z) = z^{-l_i-1}/\Gamma(\frac{3}{2}) + 0(z^{-l_i-2})$.

**) This case usually corresponds to collisions in which secondary particles with small energy are created. We assume in this case $(E - \sum_{s=1}^{N_1}\mu_s) \approx 10\,\mu_\pi$.

We can do this, because each generalized hypergeometric function is expressed as an absolutely convergent series.

Let us consider now the function

$$(26)\qquad {}_1\overline{F}_3^{(N_1)} = \sum_{k=0}^{\infty} \frac{\Gamma(2l_{N_1}+2+2k)\cdot[-\frac{1}{2}\mu R^2 (E-\sum^{N_1}\mu_s)^k]}{\Gamma(2l_{N_1}+2+2k)\cdot\Gamma(l_{N_1}+\frac{5}{2}+k)\cdot\Gamma(r_{N_1-1,N_2}+k)\cdot k!}.$$

For low energy and small R the series is alternating and strongly decreasing, so we can replace this function by the first term only, e.g., for the reaction $N+N\to n\pi$ (annihilation at rest). This approximation holds as:

$$1 \leqslant \frac{1}{2}\mu R^2\left(E-\sum^{N_1}\mu_s\right) \leqslant 6{,}5$$

and

$$r_{N_1-1,N_2} = \sum_{i=1}^{N_1}\left(l_i+\frac{3}{2}\right)+\sum_{i=1}^{N_1-1}k_i+\sum_{i=1}^{N_2}(2l_i+2+2+2h_i \gg 6{,}5\,.$$

If we use this approximation for all nonrelativistic particles we obtain:

$$(27)\qquad S_{N_1+N_2} = \sum_{l_i}^{(L)}\gamma\left(E-\sum^{N_1}\mu_s\right)^{s-1}\prod_{i=1}^{N_1}\left[\Gamma^{-1}\left(l_i+\frac{5}{2}\right)\right]\cdot$$
$$\cdot\prod_{i=1}^{N_2-1}\left\{\sum_{h_i=0}^{\infty}f_2(h_i)\cdot\left(E-\sum^{N_1}\mu_s\right)^{2h_i}\right\}\cdot {}_2F_3^{(N_2)}\,.$$

Now, for the function ${}_2F_3^{(N_2)}$ we can use Wright's asymptotic expansion, because even for smaller values of $(E-\sum^{N_1}\mu_s)$ the argument of this function is fairly large (for instance ~ 100 in the above example). So, if we apply this procedure N_2 times, we get the final formula

$$(28)\qquad S_{N_1+N_2} = \sum_{l_i}^{(L)}\alpha'\left(E-\sum^{N_1}\mu_s\right)^{s-1}\prod_{i=1}^{N_1}\beta_i\prod_{i=1}^{N_2}\gamma_i;$$

where

$$(29)\qquad \begin{cases} \alpha' = A\dfrac{Z}{\Pi(n_\nu!)}\cdot\dfrac{R^{3N_1+N_2}}{\Gamma(s')}\cdot\prod_{i=1}^{N_2}\dfrac{\mu_i}{2\pi} \\ \beta_i = R^{2l_i}\left(\dfrac{\mu}{2}\right)^{l_i+\frac{3}{2}}\cdot\Gamma^{-1}\left(l_i+\dfrac{5}{2}\right) \\ s = \sum_{i=1}^{N_1}\left(l_i+\dfrac{3}{2}\right). \end{cases}$$

In our subsequent paper, results (23) and (28) for the relative probability of transition will be applied to various physical processes, e.g. $N+N\to n\pi$, jets, showers and so on.

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INSTITUTE FOR NUCLEAR RESEARCH, POLISH ACADEMY OF SCIENCES
(INSTYTUT BADAŃ JĄDROWYCH, PAN)

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The EIH and the k-Approximation Methods

by

L. INFELD

Presented on December 9, 1960

Introduction

The two chief confirmations of General Relativity Theory are: the perihelion motion and the deflection of light. The usual way to predict these phenomena is to use a rigorous theory and rigorous equations for the one-body problem, then to solve these equations approximately, showing that the discorded terms do not change the effects forseen.

However, for a two-body problem, we use the approximation method known as the EIH Method [1]. We ask: can the known effects for the one-body problem be deduced by the EIH approximation method? The answer to this question reveals some interesting features:

1. Exactly the same effects as are deduced rigorously can be deduced by the EIH Method which seems to be restricted to small velocities.

2. The deduction is extremely simple and rigourous.

These statements will be proved in what follows, and general conclusions will be drawn.

The deduction of perihelion motion

The EIH method leads to a Lagrangian which, in the one-body case, takes the simple form

$$L = \frac{1}{2}\,\xi^a_{10'}\,\xi^a_{10'} + \frac{m}{r} - \frac{3m^2}{r^2}\,. \tag{1}$$

The metrical form ds^2 is approximately:

$$\left(\frac{ds}{dx^0}\right)^2 = \left(1 - \frac{2m}{r}\right) - \left(1 + \frac{2m}{r}\right)\frac{d\xi^a}{dx^0}\,\frac{d\xi^a}{dx^0}\,. \tag{2}$$

By ξ^a_{10} we denote the velocities and $\xi^a_{10'}$ are the conventional velocities, that is differentiated with respect to $x^{0'}$, where

$$dx^0 = \left(1 + \frac{4m}{r}\right) dx^{0'}\,. \tag{3}$$

The Latin indices run from 1 to 3. (The gravitational mass m, having the dimension of length, equals the gravitational constant times mass, divided by the square of the velocity of light.)

Therefore, we see from the Lagrangian that the problem is equivalent to a classical problem in which the potential energy

$$V = -\frac{m}{r} - \frac{3m^2}{r^2}. \tag{4}$$

As the integrals of the equations of motion we have the Law of Conservation of Momenta which in the polar co-ordinate system (φ, r) are

$$r^2 \varphi_{10'} = I \tag{5}$$

and the Law of Conservation of Energy

$$\frac{1}{2}[(r_{10'})^2 + r^2(\varphi_{10'})^2] - \frac{m}{r} - \frac{3m^2}{r^2} = E. \tag{6}$$

All this is not new [2].

Now we wish to find the perihelion motion. Eqs. (5) and (6) give us the following equation for the path, after differentiating once with respect to φ

$$\frac{d^2u}{d\varphi^2} + u\left(1 - \frac{6m^2}{I^2}\right)\frac{m}{I^2} = 0; \qquad u = \frac{1}{r}. \tag{7}$$

This equation is of a standard form; its solution is

$$u = \frac{m}{I^2}\left(1 + e\cos\left(1 - \frac{3m^2}{I^2}\right)\varphi\right) \tag{8}$$

or, putting

$$\frac{m}{I^2} = [a(1-e^2)]^{-1} \tag{9}$$

we have the well-known formula

$$r = \frac{a(1-e^2)}{1 + e\cos\left(1 - \frac{3m}{a(1-e^2)}\right)\varphi}, \tag{10}$$

which represents the perihelion motion of a planet.

The exact equation, in the same notation, is

$$\frac{d^2u}{d\varphi^2} + u = \frac{m}{I^2} + 3mu^2. \tag{11}$$

This equation can only be rigorously integrated by means of elliptic functions, yet the perihelion motion can also be obtained from the much simpler equation (7).

What happens for great energies — that is for the case $e > 1$? We then have, as in the exact equations, the conclusion

$$r = \frac{a(e^2-1)}{1 + e\cos\left(1 - \frac{3m}{a(e^2-1)}\right)\varphi}. \tag{12}$$

From this it follows that, if we neglect $3m/a\,(e^2 - 1)$, we have as in the Newtonian Theory, denoting by σ:

$$\cos\sigma = -\frac{1}{e} \tag{13}$$

that

$$-\sigma < \varphi < \sigma. \tag{14}$$

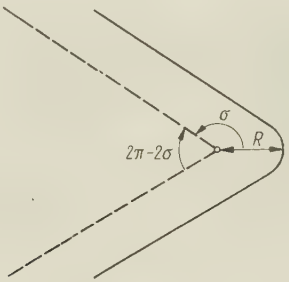
Therefore the angle between the two asymptotes is

$$2\pi - 2\sigma\,. \tag{15}$$

In the relativistic case, however, σ increases by

$$\Delta\sigma = \frac{3m\sigma}{a(e^2-1)}. \tag{16}$$

This result has an absolute meaning, since it refers to the angles between lines reaching Euclidean space.



The deflection of light

The derivation of the known formula for the deflection of light is equally simple although somewhat more tricky. The path of the light rays must be consistent with the equation

$$g_{\alpha\beta}\frac{d\xi^\alpha}{dp}\frac{d\xi^\beta}{dp} = 0\,, \tag{17}$$

where p is an arbitrary parameter for which we can take $x^{0'}$. Then, according to (3) and (2), we have in the polar co-ordinate system

$$\left(1 + \frac{4m}{r}\right) = \left(\frac{dr}{dx^{0'}}\right)^2 + r^2\left(\frac{d\varphi}{dx^{0'}}\right)^2 \tag{18}$$

and, together with (5)

$$\frac{1+4mu}{I^2} = \left(\frac{du}{d\varphi}\right)^2 + u^2\,; \qquad u = \frac{1}{r}\,. \tag{19}$$

The small expression $4m\,u$ is the relativistic expression. Without it we have

$$\frac{1}{I^2} = \left(\frac{du}{d\varphi}\right)^2 + u^2\,. \tag{20}$$

Therefore, the light-ray path is a straight line

$$u = \frac{I}{\cos \varphi} . \tag{21}$$

Let us call the distance of the light ray from the Sun R. Then we have

$$I = R. \tag{22}$$

Therefore, our equation (19), after differentiating with respect to φ, takes the form

$$\frac{2m}{R^2} = \frac{d^2 u}{d\varphi^2} + u . \tag{23}$$

The solution of this equation is

$$u = \frac{2m}{R^2}\left(1 + \frac{R}{2m}\cos \varphi\right). \tag{24}$$

Therefore, we have the famous deflection of light formula

$$2\sigma = \pi + \frac{4m}{R} . \tag{25}$$

Thus the straight line changes into a slightly curved hyperbola with the angles between the asymptotes

$$2\pi - 2\sigma = \pi - \frac{4m}{R} . \tag{26}$$

The solution of the approximate equation (23) is much simpler than the solution of the rigorous equation

$$\frac{d^2 u}{d\varphi^2} + u = 3mu^2 \tag{27}$$

which gives the same result.

Summary and conclusion

We have seen that the relativistic effects can be deduced much more simply by the New Approximation Method than by the rigorous one. The latter can only be used for the one-body problem. For the solution of the two-body problem the EIH Method was invented. What is its result? Fairly meagre compared with the cumbersome mechanism it requires. We can finally by its use reduce the two-body Lagrangian to the one-body Lagrangian [2] to obtain the same formula for the perihelion motion with the single difference that m now stands for the sum of the masses.

Recently some papers have appeared which replace the EIH method of approximation (essentially consisting in the development in power series of $1/c$) by the k-method (consisting in development with respect to the power series of the gravitational constant k). This is supposed to give new results, especially for great velocities. It may be very interesting mathematically but physically I regard it as a step backwards for the following reasons:

1. The method can only be employed (like the EIH method) for the case in which the ratio m/r is very small. In any case, we are only interested in distances for which this approximation holds since the physical radius of any body is much greater than the gravitational radius.

2. When the subscript represents the order in the EIH method we have

$$\text{(28)} \qquad \begin{cases} \xi = \underset{0}{\xi} + \underset{1}{\xi} + \underset{2}{\xi} + \cdots \\ \xi_{10} = \underset{1}{\xi_{10}} + \underset{2}{\xi_{10}} + \underset{3}{\xi_{10}} + \cdots \\ \xi_{100} = \underset{2}{\xi_{100}} + \underset{3}{\xi_{100}} + \underset{4}{\xi_{100}} + \cdots \end{cases}$$

The mass m is of the second order. Therefore in the Newtonian equations of motion the order checks on both sides of the equation. In the k-method we have

$$\text{(29)} \qquad \begin{cases} \xi = \underset{0}{\xi} + \underset{1}{\xi} + \underset{2}{\xi} + \cdots \\ \xi_{10} = \underset{0}{\xi_{10}} + \underset{1}{\xi_{10}} + \underset{2}{\xi_{10}} + \cdots \\ \xi_{100} = \underset{0}{\xi_{100}} + \underset{1}{\xi_{100}} + \underset{2}{\xi_{100}} + \cdots \end{cases}$$

Therefore, when m is of the first order, the orders in the equations of motion check only, if

$$\text{(30)} \qquad \underset{0}{\xi_{100}} = 0; \qquad \underset{1}{\xi_{100}} \neq 0,$$

that is if the initial motion is a uniform one. But we do not return to the EIH method by assuming small velocities. Thus the k-method is not a generalization of the EIH method. Indeed, according to the EIH method we should use (29) instead of (28) only for the velocities of light and not for material bodies.

3. We know that for the one-body problem the EIH method gives exactly the same result as for a rigorous solution. This we do not know for the k-method. Therefore its use for a two-body problem seems meaningless. The EIH method was invented precisely to avoid the pitfalls of the more obvious k-method.

4. The splitting of the 4-dimensional continuum into space and time is characteristic of the EIH method; it is also characteristic of all physical applications. On the contrary, however, the k-method uses only the 4-dimensional form.

INSTITUTE OF PHYSICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT FIZYKI, PAN)

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On the Asymptotical Formula for the Thermodynamical Potential of Fermi Particles

by

J. CZERWONKO

Presented by L. INFELD on December 10, 1960

We consider the Hamiltonian in the form

$$(1) \qquad H = \sum_f (\varepsilon_f - \lambda) N_f + \frac{1}{2V} \sum_{ff'} I(f,f') A_f^+ A_{f'} + \frac{1}{2V} \sum_{ff'} j(f,f') N_f N_{f'},$$

where $A_f = a_{-f} a_f$; $N_f = a_f^+ a_f$; $[a_f, a_{f'}^+]_+ = \delta_{ff'}$: and the other anti-commutators vanish. The index f denotes all states of a particle, V is the volume of the system.

We obtain the asymptotical formula for the thermodynamical potential, if V and N—total number of particles—tend to infinity in such a way that N/V remains finite. A similar formula was obtained by Bogoliubov, Zubarev and Tserkovnikov [1] for the Hamiltonian of the B.C.S. theory of superconductivity (i.e. $j(f,f') = 0$ $I(f,f') = I(\boldsymbol{k},\boldsymbol{k}')\,\sigma\sigma'$)*), and by Zubarev and Tserkovnikov [2] for a similar Hamiltonian of spinless bosons ($I(f,f') = j(f,f') = V(|\boldsymbol{k} - \boldsymbol{k}'|)$). Without loss of generality we may assume that $j(f,f')$ and $I(f,f')$ are real and symmetric functions invariant with respect to reflection of variables. We assume also that $j(f,f')$ and $I(f,f')$ vanish outside some finite domain in the momentum space.

Now, we perform the canonical transformation of Bogoliubov's type: $a_f = u_f a_f + v_f a_{\bar{f}}^+$. The canonicity conditions are: $u_f^2 + v_f^2 = 1$; $u_f = u_{-f}$; $v_f = -v_{-f}$.

We obtain

$$(2) \qquad \begin{cases} N_f = u_f^2 a_f^+ a_f - v_f^2 a_{-f}^+ a_{-f} + u_f v_f (a_f^+ a_{-f}^+ + a_{-f} a_f) + v_f^2 \\ A_f^+ = u_f^2 a_f^+ a_{-f}^+ - v_f^2 a_{-f} a_f - u_f v_f (a_f^+ a_f + a_{-f}^+ a_{-f}) + u_f v_f \equiv B_f^+ + u_f v_f \end{cases}$$

The Hamiltonian (1) may be rewritten in the form

$$H = H_0 + H' = U_0 + \sum_f H_f + \frac{1}{2v} \sum_{f,f'} H'_{ff'},$$

where

$$(3) \qquad U_0 = \sum_f \left\{ (\varepsilon_f + \xi_f - \lambda) v_f^2 + \frac{1}{2} C_f u_f v_f - \frac{1}{2} \xi_f n_f \right\},$$

*) k is the wave-vector of the particle, σ equals 1 or -1 for spin 1/2 and $-1/2$ respectively.

$$(4)\qquad H_f = [(\varepsilon_f+\xi_f-\lambda)(u_f^2-v_f^2)-2u_f v_f C_f]\,a_f^+\,a_f+ \\ +\frac{1}{2}[2u_f v_f(\varepsilon_f+\xi_f-\lambda)+C_f(u_f^2-v_f^2)](a_f^+\,a_{-f}^+ + a_{-f}\,a_f),$$

$$(5)\qquad H_{ff'} = I(f,f')\,B_f^+\,B_{f'} + j(f,f')(N_f-n_f)(N_{f'}-n_{f'}),$$

$$\xi_f = \frac{1}{V}\sum_{f'} j(f,f')\,n_{f'} \quad\text{and}\quad C_f = \frac{1}{V}\sum_{f'} I(f,f')\,u_{f'}\,v_{f'}.$$

In our consideration we introduce some parameter n_f which will be determined later. We assume, that $n_f = n_{-f}$. From this assumption, from the condition of canonicity and from the properties of functions j and J we obtain $\xi_f = \xi_{-f}$ $C_f = -C_{-f}$. It follows from [1] and [2], that the thermodynamical potential

$$\Omega_0 = -\beta^{-1}\ln \mathrm{Sp}\exp -\beta H_0$$

is asymptotically true (i.e. $\lim_{v\to\infty} v^{-1}\Omega = -\lim_{v\to\infty}(v\beta)^{-1}\ln \mathrm{Sp}\exp -\beta H$) if $\mathrm{Sp}\{B_f \exp -\beta H_0\}$ and $\mathrm{Sp}\{(N_f-n_f)\exp -\beta H_0\}$ vanish. To calculate the asymptotically true potential we must diagonalize the Hamiltonian H_0. This diagonalization can be carried out by means of Bogoliubov's canonical transformation: $a_f = g_f\beta_f + h_f\beta_{-f}^+$, where $g_f^2+h_f^2 = 1$, $g_f^v = g_{-f}$ and $h_f = -h_{-f}$. The diagonalized H_0 operator has the form

$$U + \sum_{f'} E_f\beta_f^+\beta_f,$$

where

$$U = U_0 + \frac{1}{2}\sum_{f'}[(\varepsilon_f+\xi_f-\lambda)(u_f^2-v_f^2) - 2u_f v_f C_f - E_f]$$

and

$$E_f = [(\varepsilon_f+\xi_f-\lambda)^2 + C_f^2]^{1/2}.$$

From the equation $\mathrm{Sp}\{A_f \exp -\beta H_0\} = 0$ follows

$$(6)\qquad \frac{C_f}{E_f}\,th\,\frac{\beta E_f}{2} = -2u_f v_f.$$

Equation $\mathrm{Sp}\{(N_f-n_f)\exp -\beta H_0\} = 0$ gives

$$(7)\qquad n_f = \frac{1}{2}\left[1-\frac{\varepsilon_f+\xi_f-\lambda}{E_f}\,\mathrm{th}\,\frac{\beta E_f}{2}\right].$$

From (6), (7) and from the definitions of C_f and ξ_f we obtain

$$(8)\qquad C_f = -\frac{1}{2V}\sum_{f'} J(f,f')\,\frac{C_{f'}}{E_{f'}}\,\mathrm{th}\,\frac{\beta E_{f'}}{2},$$

$$(9)\qquad \xi_f = \frac{1}{2V}\sum_{f'} j(f,f')\left[1-\frac{\varepsilon_{f'}+\xi_{f'}-\lambda}{E_{f'}}\,\mathrm{th}\,\frac{\beta E_{f'}}{2}\right].$$

Since $\mathrm{Sp}\{(N_f-n_f)\exp -\beta H_0\} = 0$, we may treat the value

$$(10)\qquad N = \sum_f n_f = \frac{1}{2}\sum_f\left\{1-\frac{\varepsilon_f+\xi_f-\lambda}{E_f}\,\mathrm{th}\,\frac{\beta E_f}{2}\right\}$$

as the total number of particles.

It may be easily checked that these three equations (8), (9), (10) are equivalent to the equation of Zubarev and Tserkovnikov [3] which was obtained by means of Bogoliubov's variational method for computing statistical sums. The asymptotically true thermodynamical potential is given by

$$(11) \qquad -\beta^{-1}\sum_f \ln(1+\exp-\beta E_f)+\frac{1}{2}\sum_f\left\{(\varepsilon_f+\xi_f-\lambda-E_f)+\right.$$
$$\left.+\frac{1}{2}\operatorname{th}\frac{\beta E_f}{2}\left[E_f-\frac{(\varepsilon_f-\lambda+\xi_f)(\varepsilon_f-\lambda)}{E_f}-\frac{\xi_f}{2}\right]\right\}.$$

We see, that the thermodynamical potential has the same dependence on the parameters ξ_f and C_f as in the variational method [3]. It may be checked that in the case of a boson Hamiltonian the variational calculation of the thermodynamic potential gives analogous results to those of Zubarev and Tserkovnikov [2], if $x \neq 0$, and to those of Valatin and Butler [4], if $x = 0$. Here x denotes a parameter occurring in the canonical transformation

$$a_k = x\delta(\boldsymbol{k})+u_k\,a_k+v_k\,a^+_{-k}; \qquad u_k^2-v_k^2=1; \qquad (u_k, v_k) = (u_{-k}, v_{-k})$$

of boson amplitudes used in the variational formalism. Eqs. (9), (10) and (11) may be solved approximately by means of the method of Zubarev and Tserkovnikov [3]. Obviously, if $j = 0$, we obtain the result of Bogoliubov, Zubarev and Tserkovnikov [1].

INSTITUTE OF THEORETICAL PHYSICS, WROCŁAW UNIVERSITY
(INSTYTUT FIZYKI TEORETYCZNEJ, UNIWERSYTET WROCŁAWSKI)

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On Gravitational Radiation Reaction Forces

by

A. SCHILD

Presented by L. INFELD on December 17, 1960

The problem of radiation in classical electrodynamics can be viewed in two ways which supplement one another. One can consider the flux of the energy tensor across distant surfaces, the resulting integrals being interpreted as energy and momentum which is radiated away to infinity [1]. Alternatively, one can consider in the equation of motion of a point charge e, the radiation reaction force, which represents the field of the charge itself, proportional to e, acting back on the charge e to give a force term proportional to e^2 [2]. The two points of view are related by considerations of energy and momentum balance.

If one attempts an analogous analysis in the general relativity theory, the first point of view presents essential difficulties. These are due to the fact that the energy tensor satisfies covariant conservation identities and to the related fact that the surface integral of the energy tensor corresponds to a sum of vectors at different points in a curved space and is thus not itself a vector. The question arises whether the second point of view can supply some information about gravitational radiation. It is the purpose of this note to show that this also presents essential difficulties.

In electrodynamics, the motion of a point charge in flat space-time is

$$m\frac{d^2\xi^\mu}{ds^2} - eF^\mu_\nu\frac{d\xi^\nu}{ds} = e^2 R^\mu\,; \qquad R^\mu = \frac{2}{3}\left(\frac{d^3\xi^\mu}{ds^3} - \frac{d\xi^\mu}{ds}\frac{d\xi_\nu}{ds}\frac{d^3\xi^\nu}{ds^3}\right), \tag{1}$$

where $\bar{F}^\mu_\nu$ is the external field, excluding the self field of the charge, and $e^2 R^\mu$ is the radiation reaction force.

In the gravitational theory, the equations of motion of particles follow from the field equations in empty space. For a test particle of small mass m, the motion is along a geodesic to the lowest order in m [3]. By going to the next higher order in m, one might expect an equation analogous to (1):

$$m\frac{\delta_0^2\xi^\mu}{\delta s_0^2} \equiv m\frac{d^2\xi^\mu}{ds_0^2} + m\left\{\begin{matrix}\mu\\ \alpha\beta\end{matrix}\right\}_0\frac{d\xi^\alpha}{ds_0}\frac{d\xi^\beta}{ds_0} = m^2 R^\mu, \tag{2}$$

where the gravitational radiation reaction force might have the form

$$R^\mu = \alpha\left(\frac{\delta_0^3\xi^\mu}{\delta s_0^3} - \frac{d\xi^\mu}{ds_0}\frac{d\xi_\nu}{ds_0}\frac{\delta_0^3\xi^\nu}{\delta s_0^3}\right), \tag{3}$$

a being a numerical constant. Here $x^\mu = \xi^\mu$ is a world line along which the gravitational field $g_{\mu\nu}(x^\varrho, m)$ has a singularity of the type characteristic of a mass pole m. The zero subscripts on ds_0 and $\left\{ {\mu \atop \alpha\beta} \right\}_0$ indicate that these quantities are evaluated by means of the background metric,

$$g_{(0)\,\mu\nu}(x^\varrho) = g_{\mu\nu}(x^\varrho, 0), \tag{4}$$

which remains regular on $x^\mu = \xi^\mu$.

A general co-ordinate transformation is of the form $x'^\mu = x'^\mu(x^\varrho, m)$. It can be considered as a combination of a transformation $x'^\mu = x'^\mu(x^\varrho)$, which does not involve the parameter m, and a transformation

$$x'^\mu = x^\mu + m\, a^\mu(x^\varrho) + m^2\, b^\mu(x^\varrho) + \ldots \tag{5}$$

R^μ is tensorial under a transformation of the first kind, but not under a transformation of the second kind. Under the transformation (5)

$$g'_{(0)\,\mu\nu}(x^\varrho) = g_{(0)\,\mu\nu}(x^\varrho), \qquad \xi'^\mu = \xi^\mu + m a^\mu(\xi^\mu) + \ldots \tag{6}$$

This means geometrically that the world line has been shifted relative to the background field by an arbitrary amount of order m. Such a shift contributes to $m\,\delta_0^2\,\xi^\mu/\delta s_0^2$ terms of order m^2 which can modify R^μ in arbitrary fashion or even wipe it out. Thus, R^μ in Eq. (2) is not tensorial and cannot be given any physical interpretation.

The argument above applies only if the subscripts 0 in Eq. (2) refer to operations involving the background metric $g_{(0)\mu\nu}$ which is obtained from the full metric $g_{\mu\nu}(x^\varrho, m)$ by simply putting $m = 0$. It does not apply, if the subscripts 0 refer to some other procedure for subtracting infinities, such as Infeld's tweedling process [4]. However, in this case and under fairly general conditions, the equation of motion of a particle is that of a geodesic to all orders in m, so that $R^\mu = 0$.

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UNIVERSITY OF TEXAS, AUSTIN, TEXAS, U.S.A.
KING'S COLLEGE, UNIVERSITY OF LONDON, U. K.

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Pionic Gauge

by

W. KRÓLIKOWSKI

Presented by W. RUBINOWICZ on December 30, 1960

It is well known that the electrodynamics is invariant to what is called the electromagnetic gauge,

$$\psi(x) \to e^{ie\Lambda(x)}\,\psi(x), \qquad A_\mu(x) \to A_\mu(x)+\partial_\mu \Lambda(x), \tag{1}$$

where $\Box\,\Lambda(x) = 0$. This invariance also guarantees zero field mass for the photon. In the case $\Lambda(x) = \text{const}$ one gets from (1) the charge gauge. The aim of the present note is to point out that in the pion mesodynamics there exists to some extent an analogy of the gauge (1), when the primary pion-pion interaction is not taken into account.

To this end let us consider the following Heisenberg equations of motion

$$(\gamma_\mu \partial_\mu + M)\,\psi = ig\gamma_5 \vec{\tau}\cdot\vec{\varphi}\psi \tag{2}$$

and

$$(\Box - \mu^2)\,\vec{\varphi} = -ig\bar{\psi}\gamma_5 \vec{\tau}\psi + \lambda\vec{\varphi}^2\,\vec{\varphi}\,. \tag{3}$$

We can easily check that these equations in the case $\lambda = 0$, but *not* the commutation relations, are invariant to the following transformation

$$\psi(x) \to e^{ig\Lambda(x)}\,\psi(x), \qquad \gamma_5\,\vec{\varphi}(x) \to \gamma_5\,\vec{\varphi}(x) + \gamma_\mu\,\partial_\mu \Lambda(x)\,\frac{\vec{\tau}}{3} \tag{4}$$

(the second relation acting on $\Psi(x)$), where $(\Box - \mu^2)\,\partial_\mu \Lambda(x) = 0$. Transformation (4) can be called the pionic gauge. In the case $\Lambda(x) = \text{const}$ transformation (4) is the well known baryon gauge.

Assuming the asymptotic condition in the ordinary form we get from (2) and (3) the following equations for the asymptotic pion fields $\vec{\varphi}^{\text{in}}_{\text{out}}(x)$:

$$[\Box - (\mu^2 + \delta\mu^2)]\,\vec{\varphi}^{\text{in}}_{\text{out}}(x) = 0. \tag{5}$$

This equation is not invariant under the pionic gauge until $\delta\mu^2 = 0$. We conclude, therefore, that in the case $\lambda = 0$ the zero field mass is attached to the pion

in the classical theory. The primary pion-pion interaction leads, however, to a non-zero field mass for the pion also in the classical theory. In the case $\lambda = 0$ the field mass of the pion is a pure quantum effect.

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INSTITUTE OF THEORETICAL PHYSICS, WARSAW UNIVERSITY
(INSTYTUT FIZYKI TEORETYCZNEJ, UNIWERSYTET WARSZAWSKI)
INSTITUTE FOR NUCLEAR RESEARCH, POLISH ACADEMY OF SCIENCES
(INSTYTUT BADAŃ JĄDROWYCH, PAN)

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Infrared Absorption Spectra of Polish Coals by Pressed Powder Method

by

L. CZUCHAJOWSKI, M. LASOŃ, W. SZYMANOWSKI, A. KUJAWSKI, I. OLSZEWSKA, A. GÓRALCZYK

Presented by W. RUBINOWICZ, on December 20, 1960

Investigations of infrared absorption spectra of coals are usually carried out by means of thin sections or powdered suspensions. The majority of authors applying these methods use Nujol mulls or potassium bromide suspensions. This last method was applied for the first time in Poland to Polish coals by W. Szymanowski and co-workers [1]. The infrared spectra of Polish coals in a Nujol mull were investigated by Urbański and co-workers [2] and by Czuchajowski [3].

The infrared absorption spectra in Nujol are, however, not complete, because Nujol shows a marked absorption in the 1400 cm.$^{-1}$ and 3000 cm.$^{-1}$ region, masking the coal sample absorption bands. There is, of course, a possibility of using hexachlorobutadiene, or other compounds having absorption spectra in different regions than Nujol and completing thus the whole spectrum of coals. But no results of such investigations were published yet for Polish coals.

The present powder method in KBr, used by us in this investigation, was proposed and described by U. Schiedt and H. Rheinwein [4], and M. M. Stimson and I. O. Donnel [5] in 1952. It permits to obtain the whole infrared spectrum in the range from 3000 cm. $^{-1}$ to 667 cm.$^{-1}$. Investigations by this method of the complete infrared spectrum of coals with increasing carbon content facilitate the comparison of spectra obtained by the KBr pressed powder method with Nujol spectra and the measurement of absorption in the range of 3000 cm.$^{-1}$ to 1400 cm.$^{-1}$. Besides, we found some regular absorption changes registered already in Nujol by Urbański and co-workers [2]. Especially in the neighbourhood of 3000 cm.$^{-1}$ absorption is interesting, because of two peaks at 3030 cm.$^{-1}$ and 2920 cm.$^{-1}$, the first of which is due to the presence of aromatic C-H bonds and the second to aliphatic [6], [7] bonds. Brown [7] has given a method for determining from these peaks the index of aromaticity of coals. Because of absorption measurements having been carried out in this band, we are able to give some additional information in this paper.

Experimental

In our present work samples of coal were ground by hand in an agate mortar. 3.5 hours grinding gives grains of a few microns in size. After very careful drying, the mixture of coal and potassium bromide was pressed into a vacuum mould. The samples were ca 1 mm. thick. The coal concentration in KBr was 0.25 per cent. Measurements were made with a semi-automatic Hilger 668 spectrometer.

Discussion

In the presented spectra of eight coals with increasing carbon content (Table I and Fig. 1) we can see the gradual decrease of absorption in all of the characteristic bands until, for the anthracitic coals, only the background absorption remains. The majority of the investigated bands are connected with characteristic oxygen containing groups. The remaining absorption bands are due to C-H and C-C bonds. Thus absorption in the 3500—3300 cm.$^{-1}$ band arises from hydrogen-bonded OH (or NH) groups. The absorption region between 1300 and 1160 cm.$^{-1}$ arises from phenolic structures and aromatic ethers and absorption bands at 1160——1060 cm.$^{-1}$ are due to C-O streching of linear and cyclic ethers and also to alcohol OH groups, and the ca 1030 cm.$^{-1}$ maximum is usually assigned to ethers of the Ph-O-CH_2 R Type. The 3030 cm.$^{-1}$ peaks are interpreted as C = H aromatic vibrations and the 2920 cm.$^{-1}$ peaks as C-H aliphatic vibrations. The 1600 cm.$^{-1}$ absorption band can be interpreted as caused by aromatic ring vibrations. It can be perhaps also connected with the presence of C-O quinonoid, or chelate systems. This interpretation of coal spectra, however right, is not final and is constantly widened.

TABLE I

Chemical composition of coal samples

No. of sample	Moisture content	Ash content	Volatile matter	Analysis % d.m.f.				by diff.
				C	H	N	S	
1	12.6	1.5	36.5	69.3	3.8	0.8	1.2	24.9
2	9.5	1.9	40.6	78.8	5.1	1.0	0.6	14.5
3	11.1	1.5	38.2	80.5	4.9	1.4	0.4	12.8
4	2.6	2.5	35.4	83.6	5.3	1.8	0.5	8.8
5	2.0	3.8	32.9	85.9	5.3	2.1	0.9	5.8
6	1.0	0.7	25.8	89.1	5.1	2.0	0.5	3.3
7	0.8	3.5	13.1	91.2	4.0	1.4	1.0	2.4
8	1.8	8.0	3.7	94.3	2.2	1.2	0.9	1.4

A series of regularities can be ascertained with respect to changes with increasing C content of absorption bands derived from reactive oxygen groups.

The 3500—3300 cm.$^{-1}$ band (with a peak near 3500 cm.$^{-1}$), which is very strong for coals with 70% C, remains still very marked in coals with 80—89% C with no conspicuous lowering of absorption, however for coals with C content higher than 90% it is no more visible. This indicates that up to the disappearence of the OH group in coal they are evident, at least partly, in a hydrogen bonded form.

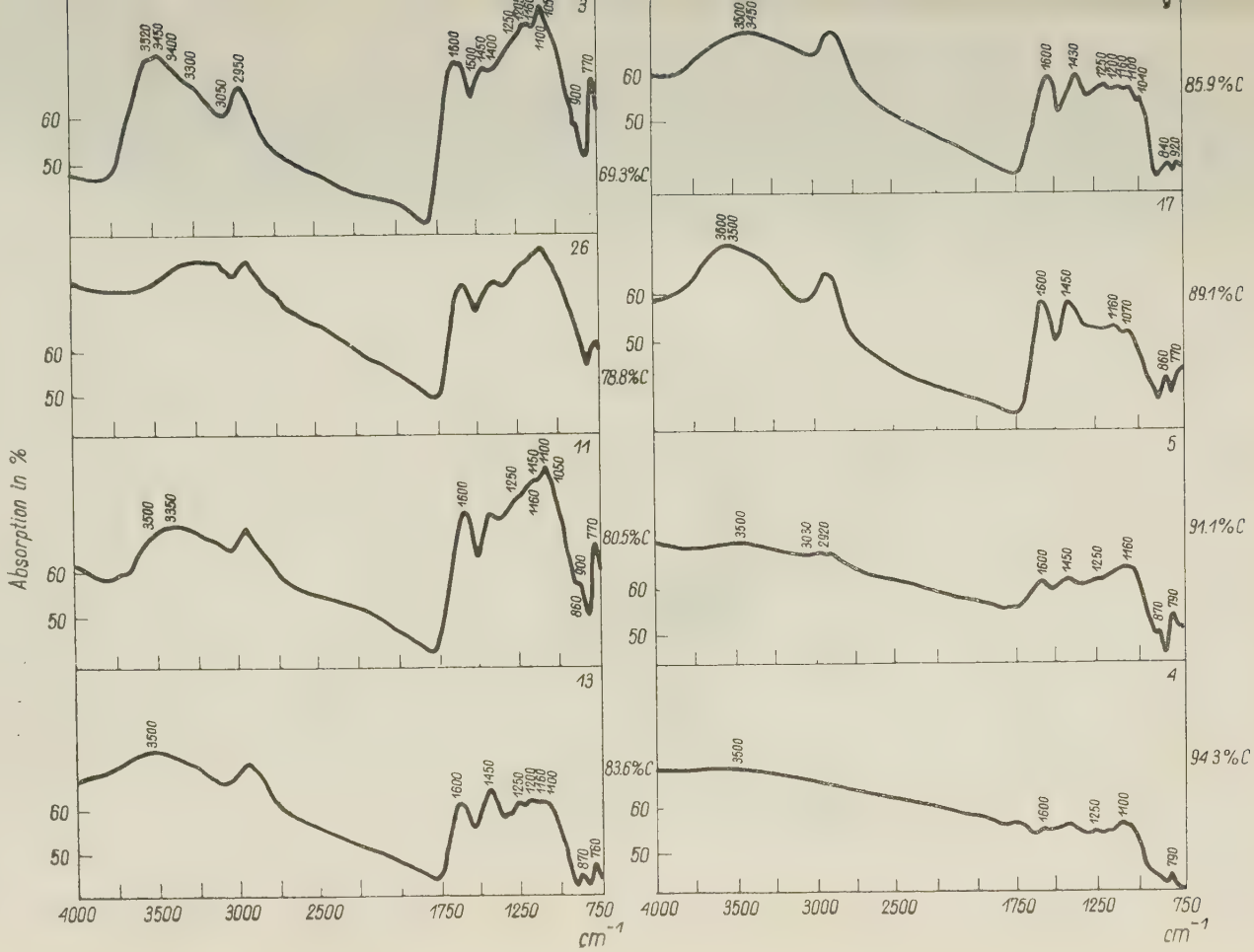

Fig. 1. Infrared absorption spectra of coal with increasing carbon content

In Nujol spectra this band (with a peak rather at ca 3300 cm.$^{-1}$) is fainter and becomes only weakly conspicuous already for coals having above 83% C.

With absorption changes in this band absorption is altered also in the next "oxygen" bands 1300 cm.$^{-1}$ 1160 cm.$^{-1}$ and 1160—1060 cm.$^{-1}$. Coals with from 70% C to 80% C have a high absorption in both bands exceeding that in the neighbouring bands at 1600 cm.$^{-1}$ and 1400 cm.$^{-1}$. In the whole range 1300—1060 cm.$^{-1}$ (peaks at 1250, 1205, 1160 and 1100 cm.$^{-1}$ are visible) absorption is steadily increasing, with decreasing wavenumbers, reaching a maximum value at 1100 cm.$^{-1}$. This indicates a marked participation not only of free phenolic OH groups, but also of oxygen in ethers (cyclic or linear). With increasing carbon content for coals having 82%—86% C, absorption in the 1160—1060 cm.$^{-1}$ band does not exceed any more its value in the 1300—1160 cm. $^{-1}$ band, and both bands are weaker than the neighbouring maxima at ca 1600 and 1400 cm.$^{-1}$. For coals with 89% C these maxima are no more visible with the exception of a 1160 cm.$^{-1}$ peak. For coals in which the carbon content exceeds 90% C the general absorption is too small to imply any definite meaning. The above mentioned bands are somewhat different in Nujol (although the same maxima can be found) for brown coal absorption in the 1300—1160 cm.$^{-1}$ and 1160—1060 cm.$^{-1}$ bands of Nujol spectrum. They are similar to those for coals with 70—80% C in KBr, but starting already from 73% are fading away for the second of these regions, namely between 1160—1060 cm.$^{-1}$. Only the 1300—1160 cm.$^{-1}$ band remains with clearly marked peaks at ca 1260 cm.$^{-1}$ and (or) near 1205 cm.$^{-1}$.

A maximum at ca 1030 cm.$^{-1}$ was not observed with the exception of the 86% C spectrum in KBr and for coals with 82% C in the Nujol spectra.

Absorption in the 1600 cm.$^{-1}$ band, which can, as already mentioned, be interpreted in many ways, decreases slowly with increasing carbon content. For coal with 91% C this maximum is still weakly conspicuous, but for coal with 94% C it is already indistinguishable from its background. Similarly absorption is decreasing in Nujol spectra. Absorption decreases almost identically in the 1500—1400 cm.$^{-1}$ band which is indiscernible in the Nujol spectra. This band is connected with aliphatic CH_2, CH_3 and cyclic CH_2 groups.

There are interesting absorption changes in the 3000 cm.$^{-1}$ band visible only KBr spectra, or in liquid substances which do not have absorption in this region, as for instance in hexachlorobutadiene. Brown [7] observed in it a sharp maximum at 2920 cm.$^{-1}$ which is visible in our spectra in a somewhat broadened form. He also found a maximum at 3030 cm.$^{-1}$ masked below 84% C by the hydrogen bonded OH band at 3300 cm.$^{-1}$. This maximum, indiscernible in our spectra, is situated in the neighbourhood of a marked absorption valley between 3500—3300 cm. $^{-1}$ and 3000 cm.$^{-1}$. Despite scrupulous investigations, required because of the small resolving power of our spectrometer in this region, we have not found for coals having 70 –89% C any more marked changes at 2920 cm. $^{-1}$. Only for coals with C content above 90% is this maximum at 2920 cm.$^{-1}$ fading away and the 3030 cm. $^{-1}$ maximum is still not conspicuous on the continuous background. Brown [7] calculated from the ratios of the optical densities of these bands, i.e. from the decimal

logarithms of the incident radiation intensity I_0 to the transmitted intensity I ($D = \log I_0/I$) the index of aromaticity according to the relation $n_{ar}/n_{al} = e_{al}/e_{ar} \times D_{3030}/D_{2920}$ where n_{ar} and n_{al} represent the number of aromatic and aliphatic C-H bonds and the e_{al} and e_{ar} values are the corresponding extinction coefficients. e_{al}/e_{ar} is taken from 4 to 3, [7].

The absence of band 3030 cm.$^{-1}$ in our absorption spectra does not permit any calculation of the index of aromaticity.

Less characteristic maxima in all coal spectra are at ca 870 cm. $^{-1}$ (most marked for coals with 83—89% C) and at 770 cm.$^{-1}$ which are both connected with single or condensed ring aromatic structures and also perhaps with mineral constituents of the investigated coals.

DEPARTMENT OF GENERAL PHYSICS "A" TECHNICAL UNIVERSITY, WARSAW
(KATEDRA FIZYKI OGÓLNEJ, "A", POLITECHNIKA WARSZAWSKA)
DEPARTMENT OF MINING CHEMISTRY, SCHOOL OF MINING AND METALLURGY, CRACOW
(KATEDRA CHEMII GÓRNICZEJ, AKADEMIA GÓRNICZO-HUTNICZA, KRAKÓW)
DEPARTMENT OF MECHANICS OF ROCK MASSES, POLISH ACADEMY OF SCIENCES
(ZAKŁAD MECHANIKI GÓROTWORU, PAN)

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EXPERIMENTAL PHYSICS

The Influence of Tartaric Acid on the Photoluminescence of Dye Solutions

by

J. GŁOWACKI

Presented by A. JABŁOŃSKI on December 5, 1960

In an earlier paper [1], some properties of tartaric acid are described. In particular, tartaric acid produces strong quenching of the photoluminescence of "yellowish" eosin (eosin potassium salt) in water. This quenching effect was generally considered to be due to non-fluorescent complexes arising between the luminophor molecule and the quencher, a conclusion derived from the finding of very considerable changes in the absorption spectrum of the luminescent substance. The intensity of luminescence light from "yellowish" eosin was found to decrease towards higher concentrations of tartaric acid, not following, however, the simple hyperbolic law as computed from the law of action of mass [2], but yielding experimental points that lie on a curve possessing a higher degree of complication. Originally this divergence was thought to arise from the luminescence of complexes. Moreover, the intensity of luminescent light was found to depend linearly on the *pH* of the solution.

The aim of the present paper was to investigate other physical properties of tartaric acid and its effect on the photoluminescence of other organic dyes. For this purpose, the author investigated the absorption spectra within the visible range and the intensity of luminescent light. The measurements were carried out photoelectrically using an FEU-19M photomultiplier. The absorption spectrum was investigated with a "Zeiss" triprismatic autocollimation spectrograph, with the modification, however, that the photographic camera was replaced by a diaphragm with slit, preceding the photomultiplier. The device was graded appropriately. The width of the spectral sector investigated was approximately 30 Å. The luminescent light intensity was measured perpendicularly to the exciting beam.

In the first place, attempts were made to come closer to the mechanism of quenching. To this aim, spectra of eosin with and without admixture of tartaric acid were investigated; in the latter case, the *pH* value was raised to its original level by adding NaOH to the solution. Aqueous solutions of eosin and of eosin with tartaric acid and NaOH exhibited no changes in the shape of the absorption spectrum. Hence it is to be concluded that a new, non-fluorescent (or practically non-fluo-

rescent) form of eosin is given rise to, as the result of a reaction between eosin and H^+ ions, tartaric acid acting as the principal agent.

Probably the most interesting results are those obtained in measuring the changes in the absorption spectrum as a function of time, at constant concentration of the luminescent ("yellowish") eosin and quenching tartatric acid substances. The spectra measured are assembled in Figs. 1, 2 and 3. From Figs. 2 and 3, the absorption

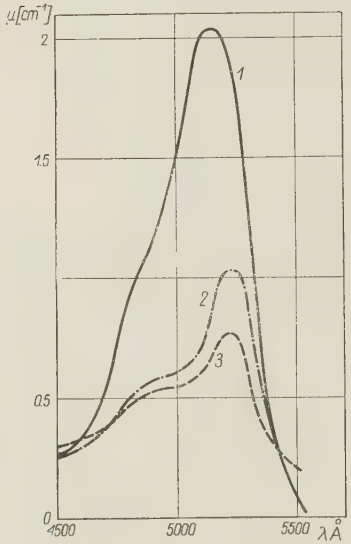

Fig. 1. Absorption spectrum of "yellowish" eosin in aqueous solution in presence of tartaric acid. Eosin concentration 10^{-5} gm./c.c. 1 — solution containing no tartaric acid; 2 — tartaric acid concentration 10^{-4} gm./c.c.; 3 — tartaric acid concentration 5×10^{-3} gm./c.c.

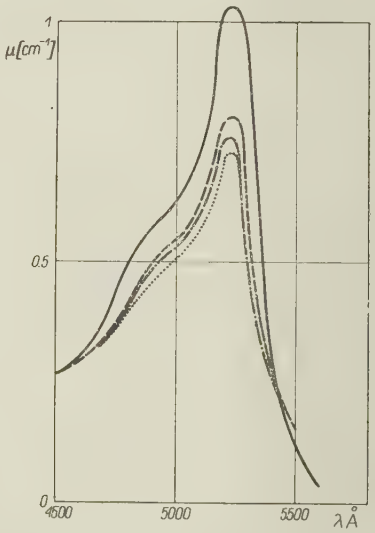

Fig. 2. Absorption spectrum of "yellowish" eosin + tartaric acid. Tartaric acid concentration 5×10^{-4} gm./c.c.
— $t = 0$, ---- $t = 22$ hours,
•—•— $t = 46$ hours,
••••• $t = 70$ hours

maximum of eosin does not undergo displacement but exhibits a decrease in height, which shows that the number of eosin molecules of the fluorescent form decreases gradually. Variation of the absorption coefficients proceeds slowly until a state of equilibrium depending on the concentration of either substance is attained. Maybe, there are as yet unknown reactions occurring between the systems at a very low rate.

The experimental conditions were as follows: Aqueous eosin solutions with and without tartaric acid were stored in the dark. In each series, eosin solutions containing no tartaric acid were also investigated for comparison. The pure eosin solution failed to exhibit any changes in the course of time, whereas the spectra of solutions containing tartaric acid changed appreciably. It would be quite incorrect to consider that this effect occurred exclusively while illuminating the solutions

during the measurements, as then the results obtained in the various tests which lasted 15 min. each and were conducted alternately from short to longer wavelengths and inversely would fail to be reproduced, whereas, as a matter of fact, the results were identical.

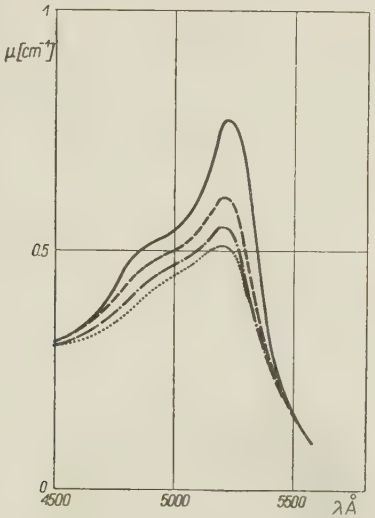

Fig. 3. Absorption spectrum of "yellowish" eosin + tartaric acid. Tartaric acid concentration 10^{-3} gm./c.c.
$t = 0$, ---- $t = 22$ hours, •—•— $t = 46$ hours, ••••• $t = 70$ hours.

The effect as a whole is better illustrated by Fig. 4 wherein the full results are assembled. The ordinates show the relative value of the quantity $A = \frac{1}{l}\int \ln \frac{I_0}{I}\, d\lambda$ as computed from graphs 2 and 3 (λ is the wavelength and I_0, I — the incident light intensity and that of the beam on emerging from an investigated solution of thickness l), whereas the abscissae bring the time. The graph was made for the maximum of eosin absorption. After a time, equilibrium is seen to set in.

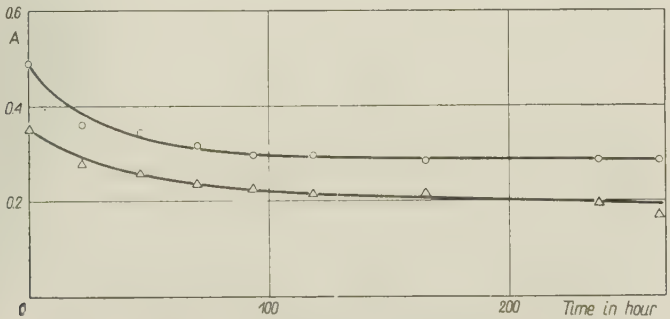

Fig. 4. Variation of absorption coefficient versus the time

Light from the visible wavelength range does not affect the luminescence of eosin; on the other hand, ultraviolet light, when used for irradiating the solution (mercury lamp), is found to diminish the intensity of luminescent light. In the present investigation no filters were employed. The solutions were contained in quartz vessels. Following 65 min. irradiation, the intensity of luminescent light of "yellowish" eosin in the presence of tartaric acid had diminished by about 70% of the intensity of a solution that had not been irradiated with a mercury lamp. The results of measurements of the ratio I/I_0 of the luminescent light intensities of irradiated and non-irradiated solutions are tabulated as dependent on the time of irradiation. The change observed would seem to arise from a perturbation in the equilibrium between the systems constituted in the present case by eosin, tartaric acid and the solvent on one hand and the solvent and eosin on the other.

TABLE

Time of irradiation in minutes	5	15	35	65
I/I_0	0.975	0.945	0.830	0.725

Fig. 5 brings the results of measurements of the absorption coefficients in "yellowish" eosin as versus the wavelength, in methyl alcohol, for different concentrations of tartaric acid. It will be noted that, in addition to a change in the absorption coefficient, the maximum of absorption is shifted slightly towards longer wavelengths. The change in absorption coefficient points to a decrease in the number of eosin molecules leading to the formation of molecules of the new, non-fluorescent form. This is corroborated by the fall in luminescent light intensity of eosin towards higher concentrations of tartaric acid (Fig. 6) and by the fact that the relative energy yield of luminescence of the solutions remains practically constant throughout the entire range of concentrations of tartaric acid investigated.

The present investigation also dealt with the dependence of the luminescent light intensity on the tartaric acid concentration in aqueous solutions of other dyes (rhodamine B, rhodamine 6G, trypaflavine and fluorescein). The luminescent light intensity of these dyes, with the exception of fluorescein, was found to undergo no changes subsequent to the introduction of tartaric acid into their solutions. This is exemplified by Fig. 6, wherein the results of such measurements in the case of rhodamine B versus the concentration of tartaric acid are plotted. Hence, it is to be concluded that the individual chemical properties of the molecules are the principal factor in the process of interaction and in determining its effect on the photoluminescence. The chemical aspect of these problems, however, did not lie within the scope of the present investigation.

The effect of tartaric acid on the fluorescence of fluorescein presents some points of interest. At very low concentrations of tartaric acid, fluorescein in aqueous and methyl alcohol solutions is quenched, whereas at higher concentrations of the acid the effect is inverse: the intensity of luminescent light increases. At a 100% con-

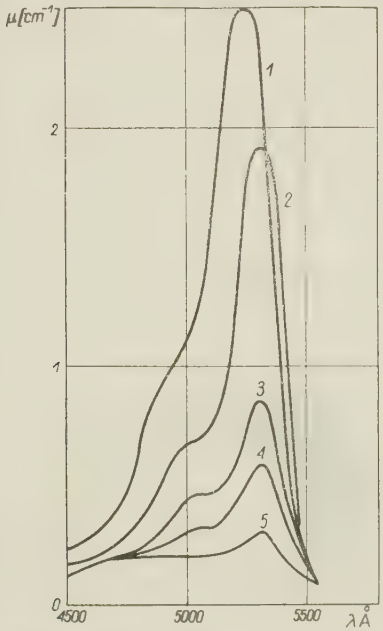

Fig. 5. Absorption of "yellowish" eosin in methyl alcohol. Eosin concentration 10 gm./c.c. 1 — no tartaric acid; 2 — tartaric acid concentration $C_w = 10^{-4}$; gm./c.c.; 3 — $C_w = 5\times10^{-4}$ gm./c.c.; 4 — $C_w = 10^{-3}$ gm./c.c.; 5 — $C_w = 5\times10^{-3}$ gm./c.c.

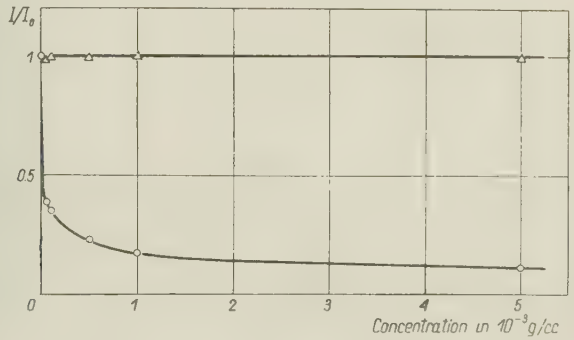

Fig. 6. Intensity of luminescence light versus concentration of tartaric acid △ rhodamine B in aqueous solution; ○ — "yellowish" eosin in methyl alcohol solution

centration of tartaric acid, in addition to fluorescence, there is long-time phosphorescence in the blue-green range. Obviously, this is the case of a rigid solution obtained by melting tartaric acid and then cooling it, subsequent to activation

with fluorescein. In the presence of tartaric acid, the absorption of fluorescein shifts towards shorter wavelengths. Fluorescein behaves like a cation, as shown in Fig. 7.

The luminescence of other dyes in pure tartaric acid was also investigated. Rhodamines were found to exhibit fluorescence, whereas no luminescence was observed in the case of eosin.

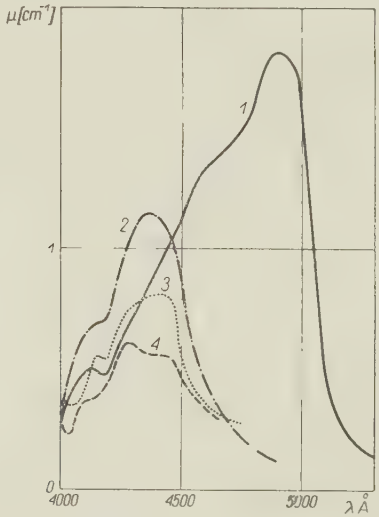

Fig. 7. Absorption spectrum of fluorescein (concentration 10^{-5} gm./c.c.) in aqueous solution + tartaric acid

1 — no tartaric acid; 2 — tartaric acid concentration $C_w = 5 \times 10^{-3}$ gm./c.c.; 3 — $C_w = 10^{-3}$ gm./c.c.; 4 — $C_w = 10^{-4}$ gm./c.c.

The results of the present investigation lead to the following conclusions:

1. Quenching of eosin by tartaric acid is a result of chemical reactions taking place in the solutions. Interaction between the molecules gives rise to non-fluorescent (or practically non-fluorescent) forms, thus diminishing the concentration of the molecules of luminescent eosin.

2. Quenching of the photoluminescence of dyes by tartaric acid depends on the internal structure and chemical properties of the molecules.

The author wishes to thank Professor A. Jabłoński for his valuable hints throughout the present investigation. The author thanks Dr J. Szychliński and Mr. T. Latowski for their discussion of the chemical aspects of the problem.

DEPARTMENT OF PHYSICS, NORMAL SCHOOL, GDAŃSK
(KATEDRA FIZYKI, WYŻSZA SZKOŁA PEDAGOGICZNA, GDAŃSK)

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EXPERIMENTAL PHYSICS

Azimuthal Angular Distribution of α Particles from the $^{10}B(n, \alpha)^7$ Li Reaction with Polarized Thermal Neutrons

by

K. MAŁUSZYŃSKA, L. NATANSON and YUAN-HAN-YUNG

Presented by M. MIĘSOWICZ on December 10, 1960

1. Anisotropy in the scattering of polarized nucleons has been for some time the subject of extensive experimental study. Until recently less work has been done on the angular distribution of the products of nuclear reactions with polarized nuclei [1]—[3].

The principle ruling these phenomena is that the angular distribution of the products of a given reaction with polarized nuclei is anisotropic, whenever nucleons from the reverse reaction at the corresponding energy are polarized and the degree of their polarization depends on angle. This well known principle has universal validity and is based on the conservation of parity in strong interactions [4], [5].

The present note concerns the simple negative case of the reaction

$$^{10}B + n \rightarrow {}^7Li + {}^4He$$

with thermal neutrons. The reverse reaction to this would be

$$^7Li + {}^4He + Q \rightarrow {}^{10}B + n$$

with $Q = 2.79$ MeV.

Owing to the centrifugal barrier, the change in angular momentum with the absorption of a thermal neutron by the ^{10}B nucleons in the $^{10}B\,(n, \alpha)\,{}^7Li$ reaction may be only $\pm 1/2\ \hbar$. The incoming neutron is bound in a *s*-state in the compound nucleus. The azimuthal angular distribution of the directions of emission of the α particles in relation to the direction of the polarization vector of the neutrons should be in this case certainly isotropic. Indeed, this applies to all nuclear reactions with thermal polarized neutrons.

2. The neutrons were polarized by transmission through polarized iron.

A neutron beam from a reactor running at 2 MW thermal power was collimated by an iron tube set into the reactor concrete shielding. The length of the tube was 90 cm., its external diameter 6 cm. and the diameter of the axial cylindrical neutron channel — 0.55 cm.

The beam then passed through a block of Armco steel 4 cm. thick. The Armco block was magnetized being inserted between the poles of an electromagnet giving a field intensity of over 13 *k*Oe.

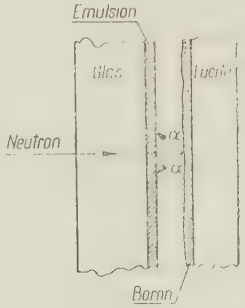

Fig. 1

The single transmission effect was measured, i.e. the increase of intensity of the transmitted neutron beam when the steel block was magnetized. The result (an increase of 23.2 ± 1.8%) is in excellent agreement with published data [6].

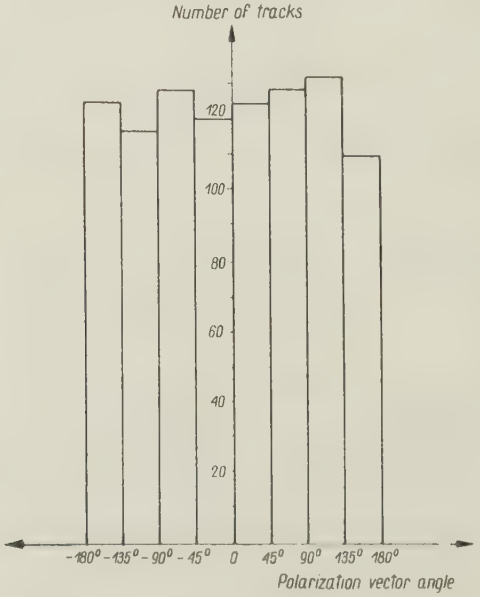

Fig. 2

Taking into account corrections for the imperfect magnetic saturation etc. we estimate the degree of polarization of the transmitted beam at about 35% for neutrons below the Cd cut-off.

3. The nuclear emulsion technique was used for the investigation of the angular distribution of the α particles. At first nuclear emulsion plates (with emulsion 100 μ thick) were impregnated with a solution of boron salt. The plate was placed perpendicular to the neutron beam. Good tracks were registered in the emulsion but it proved impossible to determine the direction in which an α particle had passed along a given path.

To avoid this difficulty non-loaded emulsions were used and pure boron powder was spread on the surface of a thin lucite plate. The boron-covered surface was held over the emulsion at a distance of about 0.05 cm. Neutrons went through the two connected plates passing on their way through glass, emulsion, boron and lucite in the order given; α particles emitted backwards, that is at angles greater than $\pi/2$ to the direction of motion of the neutrons, entered the emulsion leaving tracks of sufficient length. The α particles entered the emulsion from the surface so there could be no ambiguity as to their direction (Fig. 1).

A plate obtained in the same way but with the Armco block demagnetized, that is with non-polarized neutrons, served as control.

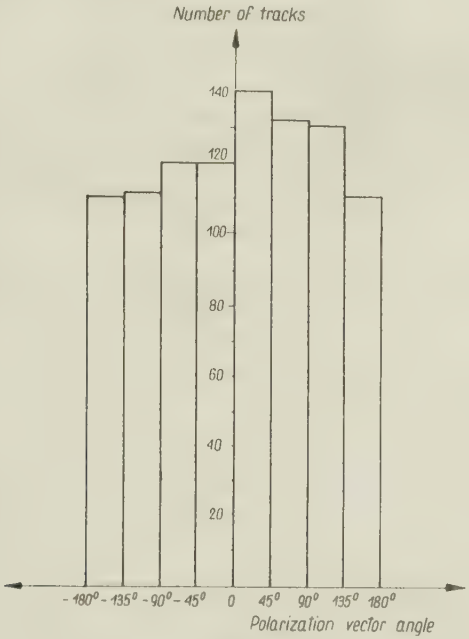

Fig. 3

Angles formed by the projections of the tracks on the emulsion plane with the direction of the neutron polarization could be easily measured with a goniometric eyepiece on the microscope. Only tracks longer than 3 μ and entering the emulsion at angles greater than $\pi/6$ to normal were counted. They were divided into groups according to the azimuthal angle of their emission in relation to the direction of the neutron polarization.

The results are illustrated on histogram I (Fig. 2) and those from the control plate on histogram II (Fig. 3).

As can be seen, no significant deviation from perfect isotropy can be detected.

The method described above for the determination of the azimuthal angle distribution of the reaction products should be suitable also for similar work with fast polarized neutrons which is being planned.

We wish to express our thanks to Z. Wilhelmi for his kind interest in our work and to J. Turkiewicz, P. Zuprański and O. Wołczek for help and assistance.

DEPARTMENT OF EXPERIMENTAL PHYSICS, UNIVERSITY, ŁÓDŹ
(KATEDRA FIZYKI DOŚWIADCZALNEJ, UNIWERSYTET, ŁÓDŹ)
INSTITUTE FOR NUCLEAR RESEARCH, POLISH ACADEMY OF SCIENCES
(INSTYTUT BADAŃ JĄDROWYCH, PAN)
INSTITUTE OF EXPERIMENTAL PHYSICS, UNIVERSITY, WARSAW
(INSTYTUT FIZYKI DOŚWIADCZALNEJ, UNIWERSYTET, WARSZAWA)

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ
И ФИЗИЧЕСКИХ НАУК

1961

ТОМ IX Резюме статей ВЫПУСК 2

В работе дается обобщение метода Чебышева на функциональные уравнения

(1) $$F(x) = 0,$$

где $F(x)$ является нелинейным функционалом в пространстве Банаха. Это является дальнейшим развитием той же идеи автора, которая позволила ему обобщить метод Ньютона на уравнения вида (1).

Работа содержит обобщение одного класса итерационных методов высших порядков на функциональные уравнения

(1) $$F(x) = 0,$$

где $F(x)$ — нелинейный функционал в пространстве Банаха. Этот класс, построенный на основании теоремы Кёнига, содержит в частности метод Ньютона и метод касательных гипербол. Обобщение является дальнейшим развитием той же идеи автора, которая позволила ему обобщить метод Ньютона на функциональные уравнения вида (1).

В настоящей работе автор доказывает точность формул перестановки (6) и (27), сингулярных итерированных интегралов в пространстве. Функция $G(z-y)$, определенная формулой (1), имеет сильную особенность для $z=y$; функция $K(x,y)$, определенная формулой (4), имеет слабую особенность для $x=y$. Функция $f(x)$ принадлежит к классу $\mathfrak{H}_\alpha^h$, это значит выполняет условия разрыва (5′) на поверхностях $S_0, S_1, \ldots, S_p$.

Настоящее сообщение является изложением результатов работы, посвященной исследованию степени однородности и универсальности пространства $\mathfrak{M}$ измеримых подмножеств прямой.

Оказывается, что $\mathfrak{M}$ — является метрически однородным по отношению к тройкам точек, но оно метрически неоднородно по отношению к четверткам. Топологически это пространство однородно, имея в виду произвольные конечные системы точек. Для всякого метрического пространства, состоящего из четырех точек в пространстве $\mathfrak{M}$ имеется изометрическое ему подпространство. $\mathfrak{M}$ не содержит некоторого пятиточечного пространства. Пространство $\mathfrak{M}$ содержит куб Гильберта.

Основным результатом работы является следующая теорема:

Теорема 2. *Если $\langle X, \varrho\rangle$ компактное, n-мерное метрическое сильно выпуклое пространство, в котором отрезки имеют однозначное продолжение и конусы над отрезком выпуклы, то X — гомеоморфно n-мерному кубу.*

Доказательства приведенных в работе теорем будут опубликованы в работе *On Convex Metric spaces* II, в журнале *Fundamenta Mathematicae.*

При определении полинома x в смысле решения уравнения $S^n x = 0$, где S — линейная операция, отображающая пространство C^1 в C^0 ($C^1 \subset C$), доказывается, что главные свойства полиномов сохраняются, если в множестве постоянных a, где $Sa = 0$, определено коммутативное умножение. Это определение является существенным обобщением определения, данного ранее.

Кроме того, рассмотрены полиномы многих переменных.

В работе произведено вычисление формул относительной вероятности перехода из состояния двух частиц с полной энергией E и полным моментом количества движения L в конечное состояние, в котором содержится число N_1 нерелятивистских и N_2 крайне релятивистских частиц с энергиями E_i и моментами количества движения l_i ($i = 1, 2, \ldots, N_1 + N_2$).

Выведенные формулы применяются в двух следующих физически интересных случаях, а именно:

(1) когда $$\left(E - \sum_{i=1}^{N_1} \mu_i\right) \geqslant 100\,\mu n$$

и

(2) когда $$\left(E - \sum_{i=1}^{N_1} \mu_i\right) \approx 10\,\mu n.$$

В работе сопоставлены два метода аппроксимации: так наз. EIH-метод и k-метод. EIH-метод состоит в развитии по отношению к степени $1/c$, тогда как k-метод — в развитии по отношению к постоянной гравитации.

EIH-меток эффективнее k-метода так как единственно он дает правильные результаты, согласующиеся с точными результатами общей теоории относительности.

В работе рассматривается асимптотически точный термодинамический потенциал для Ферми частиц, взаимодействующих посредством модельного гамильтониана более общего вида, чем в работе Боголюбова, Зубарева и Церковникова.

Доказано, что этот потенциал совпадает с термодинамическим потенциалом для общего гамильтониана системы Ферми частиц, подсчитанного вариационным методом.

В работе дискутируются уравнения движения частицы в общей теории относительности, содержащие выражения пропорциональные m^2. Эти уравнения можно считать очередной аппроксимацией (по отношению к массе m, принятой в качестве параметра развития) по сравнению с уравнениями движения пробной частицы (уравнениями геодезической линии).

Следует ожидать, что — по аналогии с электродинамикой — выражения пропорциональные m^2 описывают торможение излучением.

В работе показано, что упомянутые выражения не обладают тензорным характером и, следовательно, могут быть вытрансформированы из уравнений движения.

В сообщении указывается, что в мезодинамике π имеется некоторая аналогия калибрования, известного в электродинамике.

Исследованы спектры поглощения польских углей в инфракрасной области с возрастающим содержанием С, при применении метода прессованных суспензий в бромистом калии. Эти спектры сопоставлены со спектрами польских углей, суспендированных в парафиновом масле („Nujol“).

Большинство полученных полос обладает свойствами, характеристическим для углей, определенных при применении других методов. Однако, можно было обнаружить полосы, которые были прикрыты спектрами поглощения парафинового масла („Nujol“).

В спектрах поглощения в бромистом калии можно было обнаружить полосу 2920 $см^{-1}$ приведенную уже Брауном [7], однако, в наших спектрах нельзя было найти полосу 3030 $см^{-1}$, по всей вероятности, вследствие малой разрешающей способности нашего спектрометра. По этому поводу нельзя было установить коэффициента ароматичности.

В работе показано, что погашение калийной соли эозина винно-каменной кислотой является главным образом результатом образования новой не флуоресцирующей формы эозина. Обнаружено изменение спектров поглощения эозина при постоянии концентрации люминесцирующей и погашающей субстанции. Это изменение спектров во времени происходит очень медленно до момента достижения состояния равновесия между системами. Облучение раствора ультрафиолетовым светом вызывает нарушение состояния равновесия.

Вышеупомянутые проблемы не были исследованы с химической точки зрения.

Пучок нейтронов из реактора, поляризованных методом трансмиссии ферромагнетик применили для получения реакции $^{10}B(n,)Li$. Траектории α-частиц регистрировались в фотографической эмульсии.

Исследовано азимутальное угловое распределение направлений эмиссии α-частиц по отношению к направлению поляризации нейтронов. Согласно с предположениями теории, не констатировано значительных отклонений от изотропного распределения.

TABLE DES MATIÈRES

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